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Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICS & PHYSICAL SCIENCES

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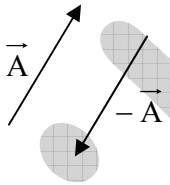
1(A).Vector Analysis

1.1 Vector Algebra

Vector quantities have both *direction* as well as *magnitude* such as velocity, acceleration, force and momentum etc. We will use $\vec{A}$ for any general vector and its magnitude by $|\vec{A}|$. In diagrams vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. Minus $\vec{A}(-\vec{A})$ is a vector with the same magnitude as $\vec{A}$ but of opposite direction.



1.1.1 Vector Operations

We define four vector operations: addition and three kinds of multiplication.

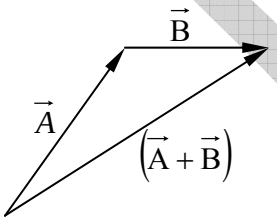
(i) Addition of two vectors

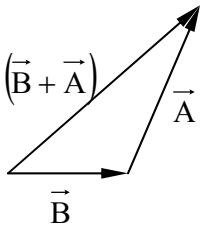
Place the tail of $\vec{B}$ at the head of $\vec{A}$; the sum, $\vec{A}+\vec{B}$, is the vector from the tail of $\vec{A}$ to the head of $\vec{B}$.

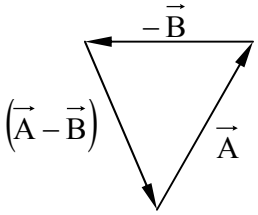
Addition is *commutative*: $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

Addition is associative: $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$

To subtract a vector, add its opposite: $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$



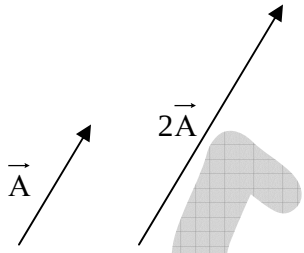




(ii) Multiplication by scalar

Multiplication of a vector by a positive scalar *a*, multiplies the *magnitude* but leaves the direction unchanged. (If *a* is negative, the direction is reversed.) Scalar multiplication is distributive:

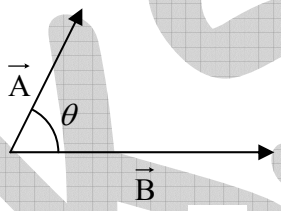
$$a\left(\vec{A}+\vec{B}\right)=a\vec{A}+a\vec{B}$$



(iii) Dot product of two vectors

The dot product of two vectors is define by

$$\vec{A}.\vec{B}=AB\cos\theta$$



where θ is the angle they form when placed tail to tail. Note that $\vec{A}.\vec{B}$ is itself a scalar. The dot product is commutative,

$$\vec{A}.\vec{B}=\vec{B}.\vec{A}$$

and distributive, $\vec{A}.\left(\vec{B}+\vec{C}\right)=\vec{A}.\vec{B}+\vec{A}.\vec{C}$.

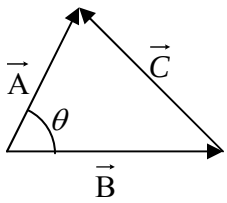
Geometrically $\vec{A}.\vec{B}$ is the product of A times the projection of $\vec{B}$ along $\vec{A}$ (or the product of B times the projection of $\vec{A}$ along $\vec{B}$).

If the two vectors are parallel, $\vec{A}.\vec{B}=AB$

If two vectors are perpendicular, then $\vec{A}.\vec{B}=0$

Law of cosines

Let $\vec{C}=\vec{A}-\vec{B}$ and then calculate dot product of $\vec{C}$ with itself.



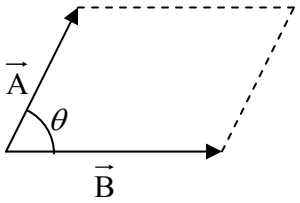
$$\vec{C}.\vec{C}=\left(\vec{A}-\vec{B}\right).\left(\vec{A}-\vec{B}\right)=\vec{A}.\vec{A}-\vec{A}.\vec{B}-\vec{B}.\vec{A}+\vec{B}.\vec{B}$$

$$C^2=A^2+B^2-2AB\cos\theta$$

(iv) Cross Product of Two Vectors

The cross product of two vectors is define by

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$$



where $\hat{n}$ is a unit vector(vector of length 1) pointing perpendicular to the plane of $\vec{A}$ and $\vec{B}$ .Of course there are two directions perpendicular to any plane "in" and "out."

The ambiguity is resolved by the **right-hand rule:**

let your fingers point in the direction of first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{n}$. (In figure $\vec{A}\times\vec{B}$ points into the page; $\vec{B}\times\vec{A}$ points out of the page)

The cross product is distributive,

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right)$$

but not commutative.

In fact $\left(\vec{B}\times\vec{A}\right)=-\left(\vec{A}\times\vec{B}\right)$.

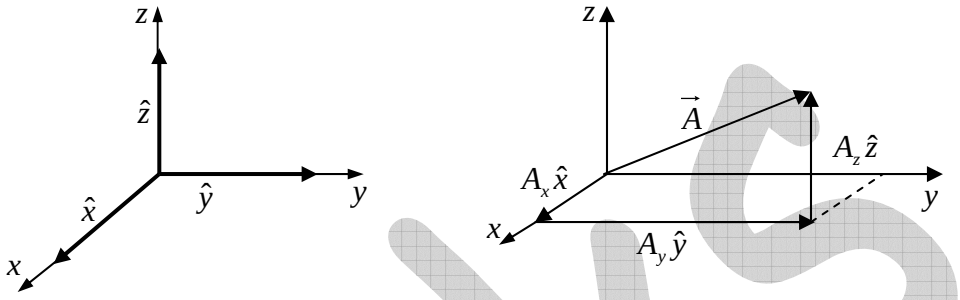
Geometrically, $\left|\vec{A}\times\vec{B}\right|$ is the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$. If two vectors are parallel, their cross product is zero.

In particular $\vec{A}\times\vec{A}=0$ for any vector $\vec{A}$

1.1.2 Vector Algebra: Component Form

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be unit vectors parallel to the *x*, *y* and *z* axis, respectively. An arbitrary vector $\vec{A}$ can be expanded in terms of these basis vectors

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$



The numbers A_x , A_y , and A_z are called component of $\vec{A}$; geometrically, they are the projections of $\vec{A}$ along the three coordinate axes.

(i) Rule: To add vectors, add like components.

$$\vec{A}+\vec{B} = \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) + \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) = \left(A_x + B_x\right)\hat{x} + \left(A_y + B_y\right)\hat{y} + \left(A_z + B_z\right)\hat{z}$$

(ii) Rule: To multiply by a scalar, multiply each component.

$$\vec{A} = \left(aA_x\right)\hat{x} + \left(aA_y\right)\hat{y} + \left(aA_z\right)\hat{z}$$

Because $\hat{x}$, $\hat{y}$ and $\hat{z}$ are mutually perpendicular unit vectors

$$\hat{x}.\hat{x} = \hat{y}.\hat{y} = \hat{z}.\hat{z} = 1;\ \ \hat{x}.\hat{y} = \hat{x}.\hat{z} = \hat{y}.\hat{z} = 0$$

Accordingly, $\vec{A}.\vec{B} = \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right).\left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) = A_x B_x + A_y B_y + A_z B_z$

(iii) Rule: To calculate the dot product, multiply like components, and add.

In particular, $\vec{A}.\vec{A} = A_x^2 + A_y^2 + A_z^2 \Rightarrow A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

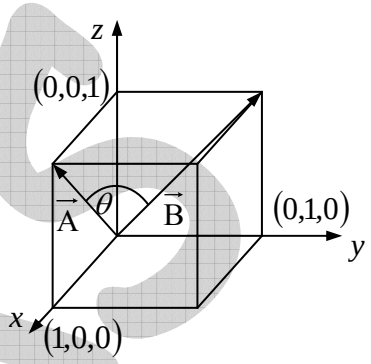
Similarly, $\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0,$

$$\hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$$
$$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x}$$
$$\hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}$$

(iv) Rule: To calculate the cross product, form the determinant whose first row is $\hat{x}$, $\hat{y}$, $\hat{z}$, whose second row is $\vec{A}$ (in component form), and whose third row is $\vec{B}$.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}$$

Example: Find the angle between the face diagonals of a cube.



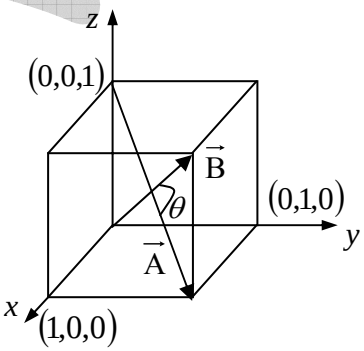
Solution: The face diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A}=1\hat{x}+0\hat{y}+1\hat{z};\quad \vec{B}=0\hat{x}+1\hat{y}+1\hat{z}$

So, $\Rightarrow \vec{A}.\vec{B}=1$

Also, $\Rightarrow \vec{A}.\vec{B}=AB\cos\theta=\sqrt{2}\sqrt{2}\cos\theta\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=60^0$

Example: Find the angle between the body diagonals of a cube.



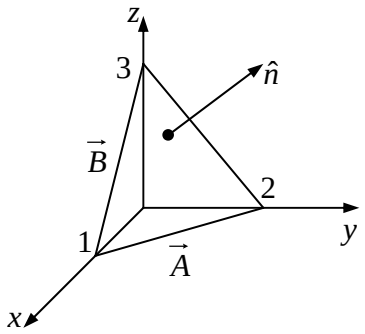
Solution: The body diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A}=\hat{x}+\hat{y}-\hat{z};\quad \vec{B}=\hat{x}+\hat{y}+\hat{z}$

So, $\Rightarrow \vec{A}.\vec{B}=1+1-1=1$

Also, $\Rightarrow \vec{A}.\vec{B}=AB\cos\theta=\sqrt{3}\sqrt{3}\cos\theta\Rightarrow\cos\theta=\frac{1}{3}\Rightarrow\theta=\cos^{-1}\left(\frac{1}{3}\right)$

Example: Find the components of the unit vector $\hat{n}$ perpendicular to the plane shown in the figure.



Solution: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

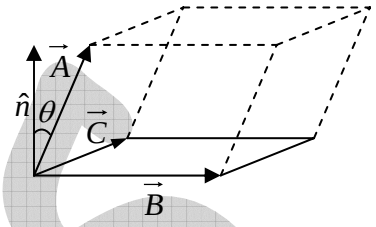
$$\vec{A}=-\hat{x}+2\hat{y};\quad \vec{B}=-\hat{x}+3\hat{z}\Rightarrow\hat{n}=\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}=\frac{6\hat{x}+3\hat{y}+2\hat{z}}{7}$$

1.1.3 Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a triple product.

(i) Scalar triple product: $\vec{A}.\left(\vec{B}\times\vec{C}\right)$

Geometrically $\left|\vec{A}.\left(\vec{B}\times\vec{C}\right)\right|$ is the volume of the parallelepiped generated by $\vec{A}$, $\vec{B}$ and $\vec{C}$, since $\left|\vec{B}\times\vec{C}\right|$ is the area of the base, and $\left|\vec{A}\cos\theta\right|$ is the altitude. Evidently,



$$\vec{A}.\left(\vec{B}\times\vec{C}\right)=\vec{B}.\left(\vec{C}\times\vec{A}\right)=\vec{C}.\left(\vec{A}\times\vec{B}\right)$$

In component form $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

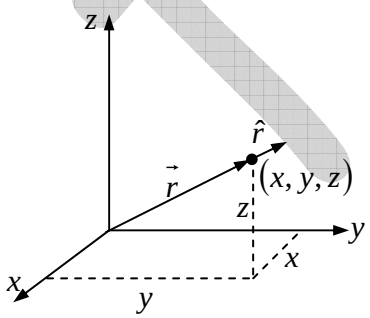
Note that the dot and cross can be interchanged: $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\times\vec{B}\right).\vec{C}$

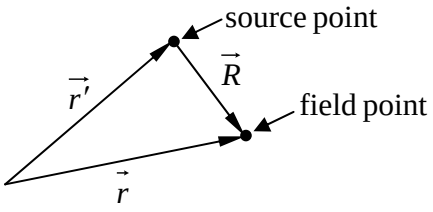
(ii) Vector triple product: $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$

The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{C}\left(\vec{A}.\vec{B}\right)$$

1.1.4 Position, Displacement, and Separation Vectors





The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin is called the ***position vector***:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}\,.$$

Its magnitude, $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin,

and $\hat{r} = \dfrac{\vec{r}}{r} = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}$ is a unit vector pointing radially outward.

The ***infinitesimal displacement vector,*** from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}\,.$$

Note: In electrodynamics one frequently encounters problems involving two points-typically, a **source point ,** $\vec{r'}$, where an electric charge is located, and a **field point,** $\vec{r}$ **,** at which we are calculating the electric or magnetic field. We can define **separation vector** from the source point to the field point by $\vec{R}$;

$$\vec{R} = \vec{r} - \vec{r'}\,.$$

Its magnitude is $R = \left|\vec{r} - \vec{r'}\right|$,

and a unit vector in the direction from $\vec{r'}$ to $\vec{r}$ is $\hat{R} = \dfrac{\vec{R}}{R} = \dfrac{\vec{r} - \vec{r'}}{\left|\vec{r} - \vec{r'}\right|}$.

In Cartesian coordinates,

$$\vec{R} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$\left|\vec{R}\right| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\hat{R} = \frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

1.2 Differential Calculus

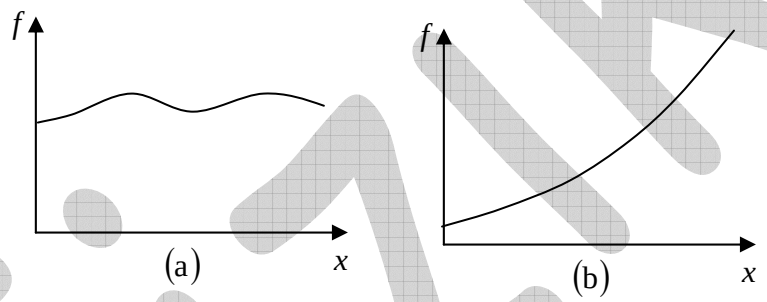
1.2.1 "Ordinary" Derivatives

Suppose we have a function of one variable: $f(x)$ then the derivative, df/dx tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right)dx$$

In words: If we change x by an amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example in figure (a), the function varies slowly with x, and the derivative is correspondingly small. In figure (b), f increases rapidly with x, and the derivative is large, as we move away from $x = 0$.

Geometrical Interpretation: The derivative df / dx is the slope of the graph of f versus x.



1.2.2 Gradient

Suppose that we have a function of three variables-say, $V(x, y, z)$ in a

$$dV = \left(\frac{\partial V}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z}\right)dz.$$

This tells us how V changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do not require an infinite number of derivatives-three will suffice: the partial derivatives along each of the three coordinate directions.

Thus $$dV = \left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)\cdot\left(dx\hat{x} + dy\hat{y} + dz\hat{z}\right) = \left(\vec{\nabla}V\right)\cdot\left(d\vec{l}\right),$$

where $\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$ is the gradient of V.

$\vec{\nabla}V$ is a vector quantity, with three components.

Geometrical Interpretation of the Gradient

Like any vector, the gradient has magnitude and direction. To determine its geometrical meaning, let's rewrite

$$dV = \vec{\nabla}V \cdot d\vec{l} = \left|\vec{\nabla}V\right|\left|d\vec{l}\right|\cos\theta$$

where θ is the angle between $\vec{\nabla}V$ and $d\vec{l}$. Now, if we fix the magnitude $\left|d\vec{l}\right|$ and search around in various directions (that is, vary θ), the maximum change in V evidently occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $\left|d\vec{l}\right|$, dT is greatest when one move in the same direction as $\vec{\nabla}V$. Thus:

The gradient $\vec{\nabla}V$ points in the direction of maximum increase of the function V.

Moreover:

The magnitude $\left|\vec{\nabla}V\right|$ gives the slope (rate of increase) along this maximal direction.

Example: Find the unit vector normal to the curve $y = x^2$ at the point (2, 4, 1).

Solution: The equation of curve in the form of surface is given by

$$x^2 - y = 0$$

A constant scalar function V on the surface is given by $V(x, y, z) = x^2 - y$

Taking the gradient

$$\vec{\nabla}V = \vec{\nabla}\left(x^2 - y\right) = \frac{\partial}{\partial x}\left(x^2 - y\right)\hat{x} + \frac{\partial}{\partial y}\left(x^2 - y\right)\hat{y} + \frac{\partial}{\partial z}\left(x^2 - y\right)\hat{z} = 2x\hat{x} - \hat{y}$$

The value of the gradient at point (2, 4, 1), $\vec{\nabla}V = 4\hat{x} - \hat{y}$

The unit vector, as required

$$\hat{n} = \pm\frac{4\hat{x} - \hat{y}}{\left|4\hat{x} - \hat{y}\right|} = \pm\frac{1}{\sqrt{17}}\left(4\hat{x} - \hat{y}\right)$$

Example: Find the unit vector normal to the surface $xy^3z^2 = 4$ at a point (-1, -1, 2).

Solution: $\vec{\nabla}\left(xy^3z^2\right) = \frac{\partial}{\partial x}\left(xy^3z^2\right)\hat{x} + \frac{\partial}{\partial y}\left(xy^3z^2\right)\hat{y} + \frac{\partial}{\partial z}\left(xy^3z^2\right)\hat{z} = \left(y^3z^2\right)\hat{x} + \left(3xy^2z^2\right)\hat{y} + \left(2xy^3z\right)\hat{z}$

At a point (-1, -1, 2), $\vec{\nabla}\left(xy^3z^2\right) = -4\,\hat{x} - 12\,\hat{y} + 4\,\hat{z}$

Unit vector normal to the surface

$$\hat{n} = \frac{-4\,\hat{x} - 12\,\hat{y} + 4\,\hat{z}}{\sqrt{(-4)^2 + \left(-12^2\right) + 4^2}} = -\frac{1}{\sqrt{11}}\left(\hat{x} + 3\,\hat{y} - \hat{z}\right)$$

1.2.3 The Operator $\vec{\nabla}$

The gradient has the formal appearance of a vector, $\vec{\nabla}$, "multiplying" a scalar *V*:

$$\vec{\nabla}V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)V$$

The term in parentheses is called "del":

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

We should say that $\vec{\nabla}$ is a vector operator that acts upon *V*, not a vector that multiplies *V*.

There are three ways the operator $\vec{\nabla}$ can act:

1. on a scalar function *V*: $\vec{\nabla}V$ (the **gradient**);
2. on a vector function $\vec{A}$, via the dot product: $\vec{\nabla}\cdot\vec{A}$ (the **divergence**);
3. on a vector function $\vec{A}$, via the cross product: $\vec{\nabla}\times\vec{A}$ (the **curl**).

1.2.4 The Divergence

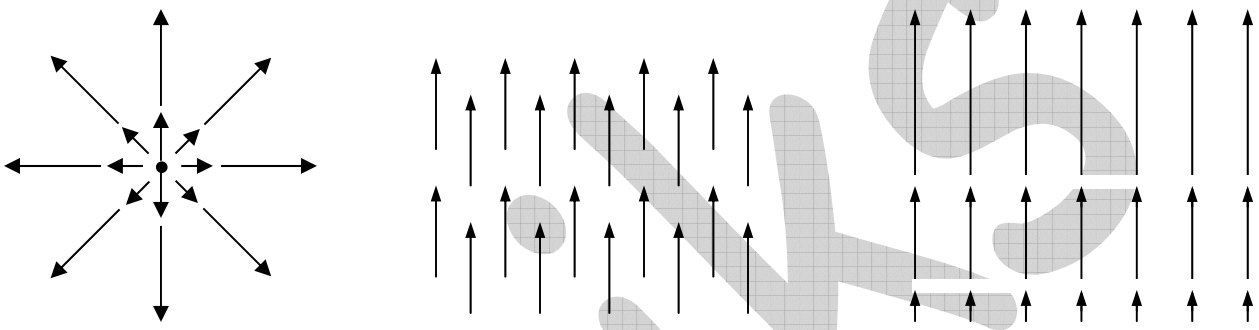
From the definition of $\vec{\nabla}$ we construct the divergence:

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Observe that the divergence of a vector function $\vec{A}$ is itself a scalar $\vec{\nabla}\cdot\vec{A}$. (You can't have the divergence of a scalar: that's meaningless.)

Geometrical Interpretation:

$\vec{\nabla}\cdot\vec{A}$ is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence. (Please understand that $\vec{A}$ here is a function-there's a different vector associated with every point in space.)



(a) (b) (c)

Example: Suppose the function sketched in above figure are $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{B} = \hat{z}$ and $\vec{C} = z\hat{z}$. Calculate their divergences.

Solution: $$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0+0+0 = 0$$

$$\vec{\nabla}\cdot\vec{C} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0+0+1 = 1$$

1.2.5 The Curl

From the definition of $\vec{\nabla}$ we construct the curl:

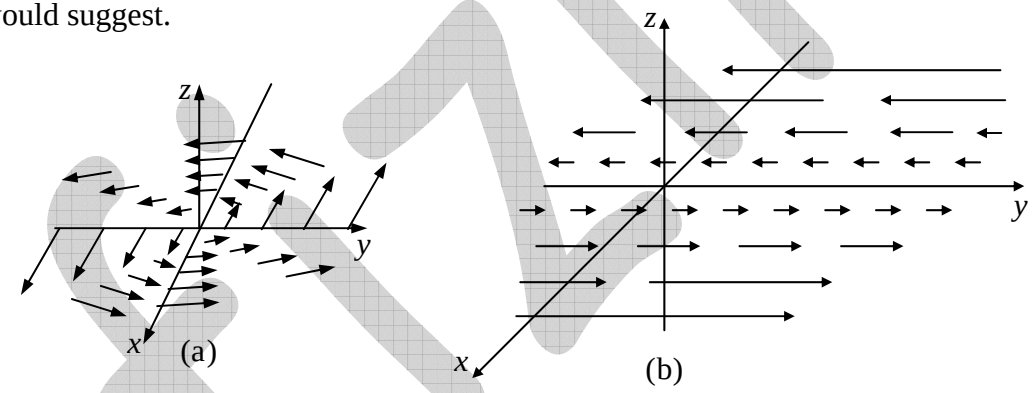
$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix}$$

$$=\hat{x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

Notice that the curl of a vector function $\vec{A}$ is, like any cross product, a vector. (You cannot have the curl of a scalar; that's meaningless.)

***Geometrical Interpretation*:**

$\vec{\nabla}\times\vec{A}$ is a measure of how much the vector $\vec{A}$ "curls around" the point in question. Figure shown below have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.



Example: Suppose the function sketched in above figure are $\vec{A}=-y\hat{x}+x\hat{y}$ and $\vec{B}=x\hat{y}$. Calculate their curls.

Solution: $\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -y & x & 0\end{vmatrix}=2\hat{z}$ and $\vec{\nabla}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & x & 0\end{vmatrix}=\hat{z}$

As expected, these curls point in the $+z$ direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"…. it just "curls around.")

Example: Given a vector function $\vec{A} = (x + c_1 z)\hat{x} + (c_2 x - 3z)\hat{y} + (x + c_3 y + c_4 z)\hat{z}$.

(a) Calculate the value of constants c_1, c_2, c_3 if $\vec{A}$ is irrotational.

(b) Determine the constant c_4 if $\vec{A}$ is also solenoidal.

(c) Determine the scalar potential function *V*, whose negative gradient equals $\vec{A}$.

Solution: If $\vec{A}$ is irrotational then, $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + c_1 z) & (c_2 x - 3z) & (x + c_3 y + c_4 z) \end{vmatrix} = 0$

$\Rightarrow \vec{\nabla} \times \vec{A} = (c_3 + 3)\hat{x} - (1 - c_1)\hat{y} + (c_2 - 0)\hat{z} = 0 \Rightarrow c_1 = 1, c_2 = 0, c_3 = -3$

(b) If $\vec{A}$ is solenoidal, $\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1 + 0 + c_4 = 0 \Rightarrow c_4 = -1$

(c) $\vec{A} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$

$\vec{A} = (x + z)\hat{x} + (-3z)\hat{y} + (x - 3y - z)\hat{z}$

$\Rightarrow \frac{\partial V}{\partial x} = -x - z \Rightarrow \mathrm{V} = -\frac{x^2}{2} - xz + f_1(y, z),$

$\frac{\partial V}{\partial y} = 3z \Rightarrow \mathrm{V} = 3yz + f_2(x, z),$

$\frac{\partial V}{\partial z} = -x + 3y + z \Rightarrow V = -xz + 3yz + \frac{z^2}{2} + f_3(x, y)$

Examination of above expressions of *V* gives a general value of

$$V = -\frac{x^2}{2} - xz + 3yz + \frac{z^2}{2}$$

1.2.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of general rules, such as the

sum rule: $$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx},$$

the rule for multiplying by a constant: $$\frac{d}{dx}(kf)=k\frac{df}{dx},$$

the product rule: $$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx},$$

and the quotient rule: $$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\dfrac{df}{dx}-f\dfrac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\vec{\nabla}(f+g)=\vec{\nabla}f+\vec{\nabla}g,\quad \vec{\nabla}\cdot\left(\vec{A}+\vec{B}\right)=\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{\nabla}\cdot\vec{B}\right),$$

$$\vec{\nabla}\times\left(\vec{A}+\vec{B}\right)=\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{\nabla}\times\vec{B}\right),$$

and

$$\vec{\nabla}(kf)=k\vec{\nabla}f,\quad \vec{\nabla}\cdot\left(k\vec{A}\right)=k\left(\vec{\nabla}\cdot\vec{A}\right),\qquad \vec{\nabla}\times\left(k\vec{A}\right)=k\left(\vec{\nabla}\times\vec{A}\right),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

fg (product of two scalar functions),

$\vec{A}\cdot\vec{B}$ (Dot product of two vectors),

and two ways to make a vector:

$f\vec{A}$ (Scalar time's vector),

$\vec{A}\times\vec{B}$ (Cross product of two vectors),

Accordingly, there are six product rules,

Two for gradients:

(i) $\vec{\nabla}(fg)=f\vec{\nabla}g+g\vec{\nabla}f,$

(ii) $\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)+\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A},$

Two for divergences:

(iii) $\vec{\nabla}\cdot\left(f\vec{A}\right)=f\left(\vec{\nabla}\cdot\vec{A}\right)+\vec{A}\cdot\left(\vec{\nabla}f\right)$,

(iv) $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$,

And two for curls:

(v) $\vec{\nabla}\times\left(f\vec{A}\right)=f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right)$,

(vi) $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)$,

It is also possible to formulate three quotient rules:

$$\vec{\nabla}\left(\frac{f}{g}\right)=\frac{g\vec{\nabla}f-f\vec{\nabla}g}{g^2},\quad \vec{\nabla}\cdot\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}g\right)}{g^2},\quad \vec{\nabla}\times\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}g\right)}{g^2}.$$

1.2.5 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\vec{\nabla}$; by applying $\vec{\nabla}$ twice we can construct five species of second derivatives. The gradient $\vec{\nabla}V$ is a vector, so we can take the *divergence* and *curl* of it:

(1) *Divergence of gradient:* $\vec{\nabla}\cdot\left(\vec{\nabla}V\right)$

$$\vec{\nabla}\cdot\left(\vec{\nabla}V\right)=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial V}{\partial x}\hat{x}+\frac{\partial V}{\partial y}\hat{y}+\frac{\partial V}{\partial z}\hat{z}\right)=\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2}.$$

This object, which we write ∇^2V for short, is called the **Laplacian** of *V*. Notice that the Laplacian of a *scalar V* is a *scalar*.

Occasionally, we shall speak of the Laplacian of a *vector*, $\nabla^2\vec{A}$. By this we mean a *vector* quantity whose *x*-component is the Laplacian of A_x, and so on:

$$\nabla^2\vec{A}\equiv\left(\nabla^2A_x\right)\hat{x}+\left(\nabla^2A_y\right)\hat{y}+\left(\nabla^2A_z\right)\hat{z}.$$

This is nothing more than a convenient extension of the meaning of ∇^2.

(2) *Curl of gradient:* $\vec{\nabla}\times\left(\vec{\nabla}V\right)$

The divergence $\vec{\nabla}\cdot\vec{A}$ is a *scalar*-all we can do is taking its gradient.

The curl of a gradient is always *zero*: $\vec{\nabla}\times\left(\vec{\nabla}V\right)=0$.

(3) *Gradient of divergence:* $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$

The curl $\vec{\nabla}\times\vec{A}$ is a *vector*, so we can take its *divergence* and *curl*.

Notice that $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$ is not the same as the Laplacian of a vector: $\nabla^2\vec{A}=\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{A}\neq\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$.

(4) *Divergence of curl:* $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)$

The divergence of a curl, like the curl of a gradient, is *always zero*:

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0.$$

(5) *Curl of curl:* $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$

As you can check from the definition of $\vec{\nabla}$:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}.$$

So curl-of-curl gives nothing new; the first term is just number (3) and the second is the Laplacian (of a vector).

1.3 Integral Calculus

1.3.1 Line, Surface, and Volume Integrals

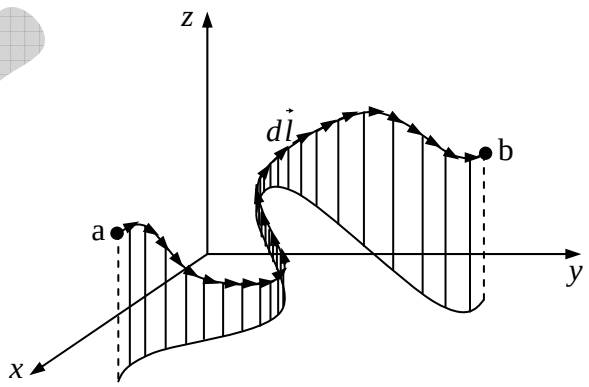
(a) Line Integrals

A line integral is an expression of the form

$$\int_a^b \vec{A}\cdot d\vec{l},$$

where $\vec{A}$ is a vector function, $d\vec{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path *P* from point *a* to point *b*. If the path in question forms a closed loop (that is, if *b* = *a*), put a circle on the integral sign:

$$\oint \vec{A}\cdot d\vec{l}.$$

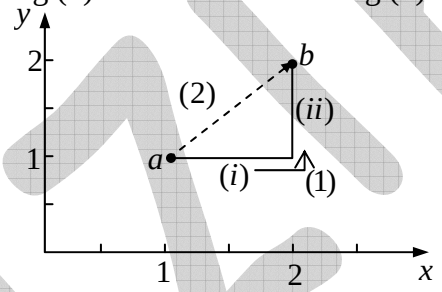


At each point on the path we take the dot product of $\vec{A}$ (evaluated at that point) with the displacement $d\vec{l}$ to the next point on the path. The most familiar example of a line integral is the work done by a force $\vec{F}$:

$$W = \int \vec{F} \cdot d\vec{l}$$

Ordinarily, the value of a line integral depends critically on the particular path taken from *a* to *b*, but there is an important special class of vector functions for which the line integral is independent of the path, and is determined entirely by the end points(A *force* that has this property is called ***conservative***.)

Example: Calculate the line integral of the function $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point *a* = (1, 1, 0) to the point *b* = (2, 2, 0), along the paths (1) and (2) as shown in figure. What is $\oint \vec{A} \cdot d\vec{l}$ for the loop that goes from *a* to *b* along (1) and returns to *a* along (2)?



Solution: Since $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Path (1) consists of two parts. Along the "horizontal" segment $dy = dz = 0$, so

(i) $d\vec{l} = dx\hat{x}, \quad y = 1, \quad \vec{A} \cdot d\vec{l} = y^2 dx = dx,$ so $\int \vec{A} \cdot d\vec{l} = \int_1^2 dx = 1$

On the "vertical" stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\hat{y}, \quad x = 2, \quad \vec{A} \cdot d\vec{l} = 2x(y+1)dy = 4(y+1)dy,$ so $\int \vec{A} \cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10.$

By path (1), then,

$$\int_a^b \vec{A} \cdot d\vec{l} = 1 + 10 = 11$$

Meanwhile, on path (2) $x = y$, $dx = dy$, and $dz = 0$, so

$$d\vec{l} = dx\hat{x} + dx\hat{y}, \quad \vec{A}\cdot d\vec{l} = x^2dx + 2x(x+1)dx = (3x^2 + 2x)dx$$

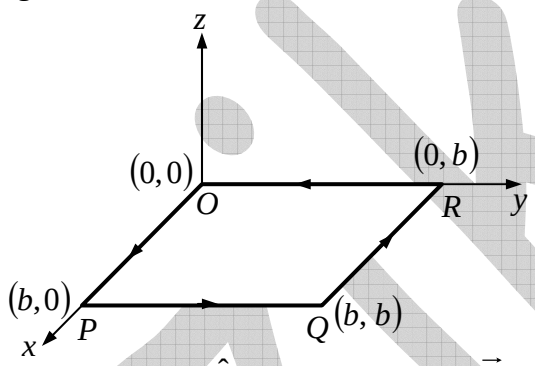
so

$$\int_a^b \vec{A}\cdot d\vec{l} = \int_1^2 (3x^2 + 2x)dx = \left(x^3 + x^2\right)\Big|_1^2 = 10$$

For the loop that goes out (1) and back (2), then,

$$\oint \vec{A}\cdot d\vec{l} = 11 - 10 = 1$$

Example: Find the line integral of the vector $\vec{A} = (x^2 - y^2)\hat{x} + 2xy\hat{y}$ around a square of side '*b*' which has a corner at the origin, one side on the *x* axis and the other side on the *y* axis.



Solution: In a Cartesian coordinate system $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, $\vec{A} = (x^2 - y^2)\hat{x} + 2xy\hat{y}$

$$\oint_{OPQRO} \vec{A}\cdot d\vec{l} = \oint_{OPQRO} \left[(x^2 - y^2)dx + 2xydy\right]$$

Along OP, $y = 0, dy = 0 \Rightarrow \int_{OP} \vec{A}\cdot d\vec{l} = \int_{x=0}^{b} x^2dx = \frac{b^3}{3}$

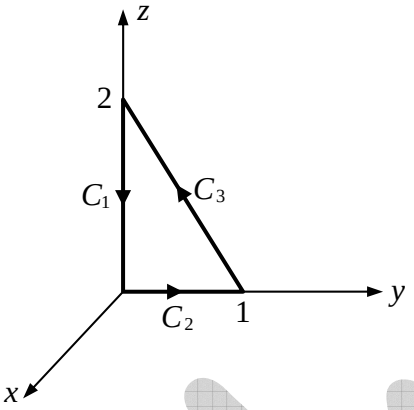
Along PQ, $x = b, dx = 0 \Rightarrow \int_{PQ} \vec{A}\cdot d\vec{l} = \int_{y=0}^{b} 2b\,y\,dy = b^3$

Along QR, $y = b, dy = 0 \Rightarrow \int_{QR} \vec{A}\cdot d\vec{l} = \int_{x=b}^{0} (x^2 - b^2)dx = \left(\frac{x^2}{3} - b^2x\right)\Big|_{x=b}^{0} = \frac{2}{3}b^3$

Along RO, $x = 0, dx = 0 \Rightarrow \int_{RO} \vec{A}\cdot d\vec{l} = 0$

$$\oint \vec{A}\cdot d\vec{l} = \int_{OP} \vec{A}\cdot d\vec{l} + \int_{PQ} \vec{A}\cdot d\vec{l} + \int_{QR} \vec{A}\cdot d\vec{l} + \int_{RO} \vec{A}\cdot d\vec{l} = \frac{b^3}{3} + b^2 + \frac{2}{3}b^3 + 0 = 2b^3$$

Example: Compute the line integral $\vec{F} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$ along the triangular path shown in figure.



Solution: Line Integral $\oint \vec{F}.d\vec{l} = \int_{C_1}\vec{F}.d\vec{l} + \int_{C_2}\vec{F}.d\vec{l} + \int_{C_3}\vec{F}.d\vec{l}$

On path C_1, $x = 0,\ y = 0,\ d\vec{l} = dz\hat{z}$

$$\int_{C_1}\vec{F}\cdot d\vec{l} = \int_{z=2}^{0}\left[6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}\right]\cdot dz\hat{z} = \int_{z=2}^{0} z\,dz = \frac{z^2}{2}\Big|_2^0 = -\frac{4}{2} = -2$$

On path C_2, $x = 0,\ z = 0,\ d\vec{l} = dy\hat{y} \Rightarrow \int_{C_2}\vec{F}\cdot d\vec{l} = \int_{y=0}^{1} yz^2\,dy = 0$

On path C_3 the slope of line is-2 and intercept on z axis is $2 \Rightarrow z = -2y + 2 = 2(1-y)$

and the connecting points are (0, 1, 0) and (0, 0, 2)

On C_3, $x=0,\ dx = 0 \qquad d\vec{l} = dy\hat{y} + dz\hat{z}$

$$\int_{C_3}\vec{F}\cdot d\vec{l} = \int_{C_3}(yz^2)dy + (3y+z)dz = \int_{y=1}^{0} y[2(1-y)]^2\,dy + \int_{z=0}^{2}\left[3\left(\frac{2-z}{2}\right) + z\right]dz$$

$$= \int_1^0 \left(4y + 4y^3 - 8y^2\right)dy + \int_{z=0}^{2}\left(3 - \frac{z}{2}\right)dz = 4\frac{y^2}{2}\Big|_1^0 + \frac{4y^4}{4}\Big|_1^0 - \frac{8y^3}{3}\Big|_1^0 + 3z\Big|_0^2 - \frac{1}{2}\frac{z^2}{2}\Big|_0^2$$

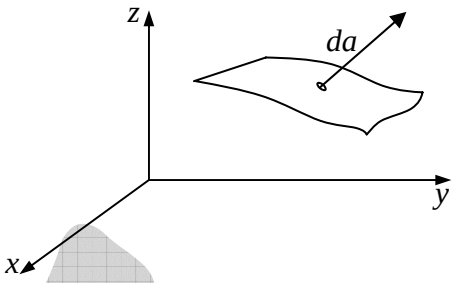
$$= -2 - 1 + \frac{8}{3} + 6 - 1 = \frac{14}{3}$$

$$\oint \vec{F}.d\vec{l} = \int_{C_1}\vec{F}.d\vec{l} + \int_{C_2}\vec{F}.d\vec{l} + \int_{C_3}\vec{F}.d\vec{l} = -2 + 0 + \frac{14}{3} = \frac{8}{3}$$

(b) Surface Integrals

A surface integral is an expression of the form

$$\int_S \vec{A}\cdot d\vec{a}$$

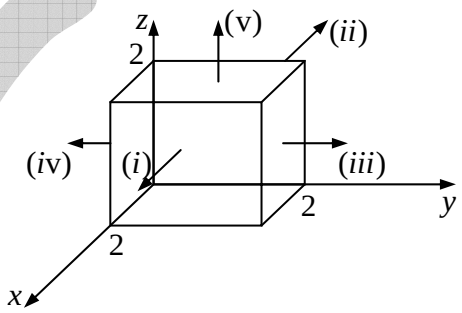


where $\vec{A}$ is again some vector function, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface(as shown in figure). There are, of course, two directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous. If the surface is closed then "outward" is positive, but for open surfaces it's arbitrary.

If $\vec{A}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{A}\cdot d\vec{a}$ represents the total mass per unit time passing through the surface-hence the alternative name, "flux."

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface, and is determined entirely by the boundary line.

Example: Calculate the surface integral of $\vec{A} = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) as shown in figure. Let "upward and outward" be the positive direction, as indicated by the arrows.



Solution: Taking the sides one at a time:

(i) $x = 2,\ d\vec{a} = dydz\hat{x},\ \vec{A}\cdot d\vec{a} = 2xzdydz = 4zdydz,$ so $\int \vec{A}\cdot d\vec{a} = 4\int_0^2 dy\int_0^2 zdz = 16.$

(ii) $x = 0,\ d\vec{a} = -dydz\hat{x},\ \vec{A}\cdot d\vec{a} = -2xzdydz = 0,$ so $\int \vec{A}\cdot d\vec{a} = 0.$

(iii) $y = 2,\ d\vec{a} = dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = (x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 (x+2)dx\int_0^2 dz = 12.$

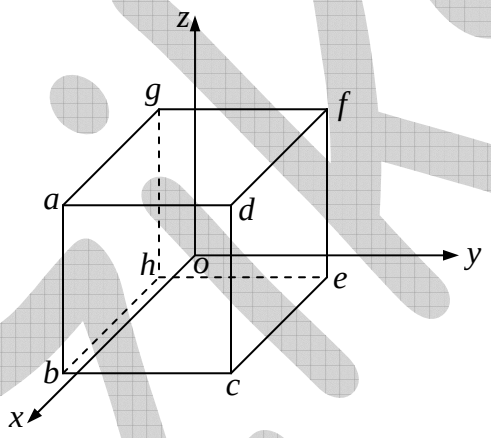
(iv) $y = 0,\ d\vec{a} = -dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = -(x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = -\int_0^2 (x+2)dx\int_0^2 dz = -12$.

(v) $z = 2,\ d\vec{a} = dx\,dy\,\hat{z},\ \vec{A}\cdot d\vec{a} = y(z^2-3)dx\,dy = y\,dx\,dy,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 dx\int_0^2 ydy = 4$

Evidently the total flux is

$$\int_{\text{surface}} \vec{A}\cdot d\vec{a} = 16+0+12-12+4 = 20$$

Example: Given a vector $\vec{A} = (x^2-y^2)\hat{x} + 2xy\hat{y} + (x^2-xy)\hat{z}$. Evaluate $\oint_S \vec{A}\cdot d\vec{a}$ over the surface of the cube with the centre at the origin and length of side '*a*'.



Solution: The surface integral is performed on all faces. The differential surface on the different faces are $\pm dy\,dz\hat{x}, \pm dx\,dz\hat{y},$ and $\pm dx\,dy\hat{z}$

Face *abcd*, $x = \frac{a}{2}$

$$\int_{abcd} \vec{A}\cdot d\vec{a} = \int_S \left[(x^2-y^2)\hat{x} + 2xy\hat{y} + (x^2-xy)\hat{z}\right]\cdot\left[dy\,dz\hat{x}\right] = \int_{y=-a/2}^{+a/2}\ \int_{z=-a/2}^{a/2} (x^2-y^2)dydz = \frac{a^4}{4}$$

Face *efgh*, $x = -\frac{a}{2}$

$$\Rightarrow \int_{efgh} \vec{A}\cdot d\vec{a} = \int_{efgh} \vec{A}\cdot(-dy\,dz\hat{x}) = \int_{y=-a/2}^{a/2}\ \int_{z=-a/2}^{a/2} -(x^2-y^2)dy\,dz = -\frac{a^4}{4}$$

Face *cdfe*, $y = \frac{a}{2}$

$$\Rightarrow \int_{cdfe} \vec{A}\cdot d\vec{a} = \int_S \vec{A}\cdot dx\,dz\hat{y} = \int_{x=-a/2}^{a/2}\int_{z=-a/2}^{a/2} 2xy\,dx\,dz = 0$$

Face *aghb*, $y = -\frac{a}{2}$

$$\Rightarrow \int_{aghb} \vec{A}\cdot d\vec{a} = \int_{x=-a/2}^{a/2}\int_{z=-a/2}^{a/2} \vec{A}\cdot\left(-dx\,dz\hat{y}\right) = 0$$

Similarly for the other two faces *adfg* and *bceh* we can find the surface integral with $d\vec{a} = \pm dx\;\; dy\hat{z}$, respectively. The addition of these two surface integrals will be zero.

In the present case sum of all the surface integral

$$\oint_S \vec{A}.d\vec{a} = 0$$

(c) Volume Integrals

A volume integral is an expression of the form

$$\int_V T\,d\tau,$$

where *T* is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx\,dy\,dz.$$

For example, if *T* is the density of a substance (which might vary from point to point) then the volume integral would give the total mass. Occasionally we shall encounter volume integrals of vector functions:

$$\int \vec{A}\,d\tau = \int \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)d\tau = \hat{x}\int A_x\,d\tau + \hat{y}\int A_y\,d\tau + \hat{z}\int A_z\,d\tau\,;$$

because the unit vectors are constants, they come outside the integral.

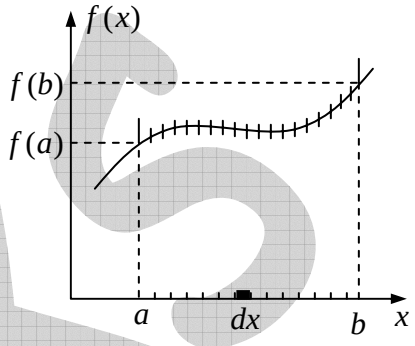
1.3.2 The Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable. The **fundamental theorem of calculus** states:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a) \text{ or } \int_a^b F(x)dx = f(b) - f(a)$$

where $df/dx = F(x)$.

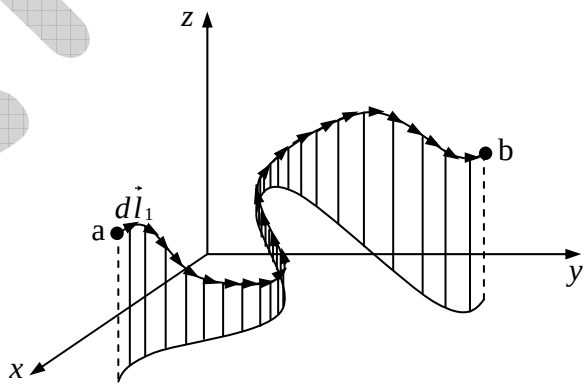
***Geometrical Interpretation*:**



According to equation $df = (df/dx)\,dx$ is the infinitesimal change in f when one goes from (x) to $(x + dx)$. The fundamental theorem says that if you chop the interval from a to b into many tiny pieces, dx, and add up the increments df from each little piece, the result is equal to the total change in f: $f(b) - f(a)$.

In other words, there are two ways to determine the total change in the function: either subtract the values at the ends or go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

1.3.3 The Fundamental Theorem for Gradients



Suppose we have a scalar function of three variables $V(x, y, z)$. Starting at point a, we moves a small distance $d\vec{l}_1$. Then

$$dV = \left(\vec{\nabla}V\right)\cdot d\vec{l}_1 .$$

Now we move a little further, by an additional small displacement $d\vec{l}_2$; the incremental change in V will be $\left(\vec{\nabla}V\right)\cdot d\vec{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point b. At each step we compute the gradient of V (at that point) and dot it into the displacement $d\vec{l}$ …this gives us the change in V. Evidently the total change in V in going from a to b along the path selected is

$$\int_{a\,P}^{b}\left(\vec{\nabla}V\right)\cdot d\vec{l} = V(b) - V(a).$$

This is called the fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (*a* and *b*).

Geometrical Interpretation:

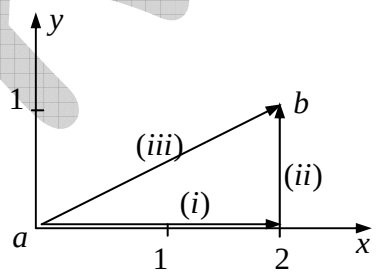
Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way (that's the fundamental theorem).

Corollary 1: $\int_a^b \left(\vec{\nabla} V\right)\cdot d\vec{l}$ is independent of path taken from *a* to *b*.

Corollary 2: $\oint \left(\vec{\nabla} V\right)\cdot d\vec{l} = 0$, since the beginning and end points are identical, and hence

$$V(b) - V(a) = 0.$$

Example: Let $V = xy^2$, and take point *a* to be the origin (0, 0, 0) and *b* the point (2, 1, 0). Check the fundamental theorem for gradients.



Solution: Although the integral is independent of path, we must pick a specific path in order to evaluate it. Let's go out along the *x* axis (step *i*) and then up (step *ii*). As always

$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z},\ \vec{\nabla} V = y^2\hat{x} + 2xy\,\hat{y}$

(i) $y = 0;\ d\vec{l} = dx\,\hat{x},\ \vec{\nabla} V \cdot d\vec{l} = y^2 dx = 0,$ so $\int_i \vec{\nabla} V \cdot d\vec{l} = 0$

(ii) $x = 2;\ d\vec{l} = dy\,\hat{y},\ \vec{\nabla} V . d\vec{l} = 2xy dy = 4y\,dy,$ so $\int_{ii} \vec{\nabla} V \cdot d\vec{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$

Evidently the total line integral is 2.

This consistent with the fundamental theorem: $T(\mathrm{b}) - T(\mathrm{a}) = 2 - 0 = 2$.

Calculate the same integral along path (iii) (the straight line from *a* to *b*):

(iii) $y = \frac{1}{2}x, dy = \frac{1}{2}dx, \vec{\nabla} V \cdot d\vec{l} = y^2 dx + 2xy\,dy = \frac{3}{4}x^2 dx,$ so

$$\int_{iii} \vec{\nabla} V \cdot d\vec{l} = \int_0^2 \frac{3}{4}x^2\,dx = \frac{1}{4}x^3\Big|_0^2 = 2$$

Thus the integral is independent of path.

1.3.4 The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oint_S \vec{A}\cdot d\vec{a}$$

This theorem has at least three special names: ***Gauss's theorem, Green's theorem***, or, simply, the ***divergence theorem***. Like the other "fundamental theorems," it says that the integral of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume). Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a (closed) surface.

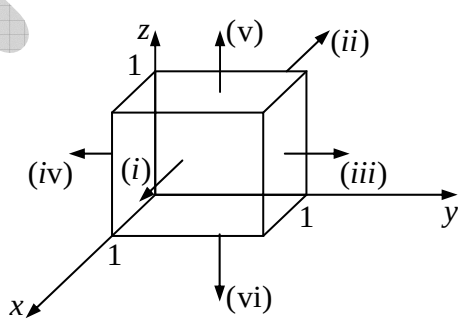
Geometrical Interpretation:

If $\vec{A}$ represents the flow of an incompressible fluid, then "the flux of $\vec{A}$ (the right side of equation) is the total amount of fluid passing out through the surface, per unit time and the left side of equation shows an equal amount of liquid will be forced out through the boundaries of the region.

Example: Check the divergence theorem using the function

$$\vec{A} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$$

and the unit cube situated at the origin.



Solution: In this case

$$\vec{\nabla}\cdot\vec{A} = 2(x+y),$$

and $$\int_V 2(x+y)d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz,$$

$$\int_0^1 (x+y)dx = \frac{1}{2} + y, \int_0^1\left(\frac{1}{2}+y\right)dy = 1, \int_0^1 1dz = 1.$$

Evidently, $$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 2$$

To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 y^2\,dy\,dz = \frac{1}{3}$ (ii) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 y^2\,dy\,dz = -\frac{1}{3}$

(iii) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 \left(2x+z^2\right)dx\,dz = \frac{4}{3}$ (iv) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\frac{1}{3}$

(v) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 2y\,dx\,dy = 1$ (vi) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 0\,dx\,dy = 0$

So the total flux is:

$$\oint_S \vec{A}\cdot d\vec{a} = \frac{1}{3}-\frac{1}{3}+\frac{4}{3}-\frac{1}{3}+1+0 = 2\,.$$

1.3.5 The Fundamental Theorem for Curls

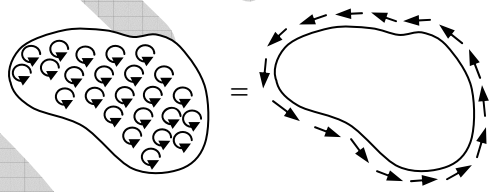
The fundamental theorem for curls, which goes by the special name of ***Stokes' theorem***, states that

$$\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint_P \vec{A}\cdot d\vec{l}$$

As always, the integral of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary (here, the perimeter of the patch). As in the case of the divergence theorem, the boundary term is itself an integral-specifically, a closed line integral.

Geometrical Interpretation:

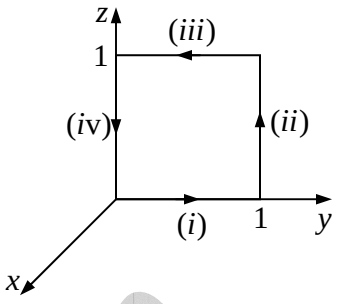
The integral of the curl over some surface (or, more precisely, the flux of the curl through that surface) represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (as shown in figure).



Corollary 1: $\int \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of equation vanishes.

Example: Suppose $\vec{A} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.



Solution: Here

$\vec{\nabla} \times \vec{A} = (4z^2 - 2x)\hat{x} + 2z\,\hat{z}$ and $d\vec{a} = dy\,dz\,\hat{x}$

(In saying that $d\vec{a}$ points in the x direction, we are choosen to a counterclockwise line integral. We could as well write $d\vec{a} = -dy\,dz\,\hat{x}$, but then we have to go clockwise.) Since $x = 0$ for this surface,

$$\int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_0^1 \int_0^1 4z^2\, dy\, dz = \frac{4}{3}$$

Now, what about the line integral? We must break this up into four segments:

(i) $x = 0, \quad z = 0, \quad \vec{A} \cdot d\vec{l} = 3y^2 dy, \qquad \int \vec{A} \cdot d\vec{l} = \int_0^1 3y^2 dy = 1,$

(ii) $x = 0, \quad y = 1, \quad \vec{A} \cdot d\vec{l} = 4z^2 dz, \int \vec{A} \cdot d\vec{l} = \int_0^1 4z^2 dz = \frac{4}{3},$

(iii) $x = 0, \quad z = 1, \quad \vec{A} \cdot d\vec{l} = 3y^2 dy, \qquad \int \vec{A} \cdot d\vec{l} = \int_1^0 3y^2 dy = -1,$

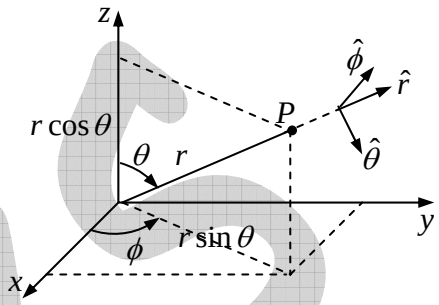
(iv) $x = 0, \quad y = 0, \quad \vec{A} \cdot d\vec{l} = 0, \qquad \int \vec{A} \cdot d\vec{l} = \int_1^0 0\,dz = 0,$

So $\oint \vec{A} \cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$

1.4 Curvilinear Coordinates

1.4.1 Spherical Polar Coordinates

The spherical polar coordinates (r,θ,ϕ) of a point *P* are defined in figure shown below; *r* is the distance from the origin (the magnitude of the position vector), θ (the angle down from the *z* axis) is called the polar angle, and ϕ (the angle around from the *x* axis) is the azimuthal angle. Their relation to Cartesian coordinates (*x*, *y*, *z*) can be read from the figure:



$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

and $r = \sqrt{x^2+y^2+z^2}$, $\theta = \cos^{-1}\dfrac{z}{r}$, $\phi = \tan^{-1}\dfrac{y}{x}$

Figure shows three unit vectors $\hat{r},\hat{\theta},\hat{\phi}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{x},\hat{y},\hat{z}$), and any vector $\vec{A}$ can be expressed in terms of them in the usual way:

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

A_r, A_θ, and A_ϕ are the radial, polar and azimuthal components of $\vec{A}$.

An infinitesimal displacement in the $\hat{r}$ direction is simply *dr* (figure *a*), just as an infinitesimal element of length in the *x* direction is *dx*:

$$dl_r = dr$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (figure *b*) is $r\,d\theta$

$$dl_\theta = r\,d\theta$$

Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (figure *c*) is $r\sin\theta\,d\phi$

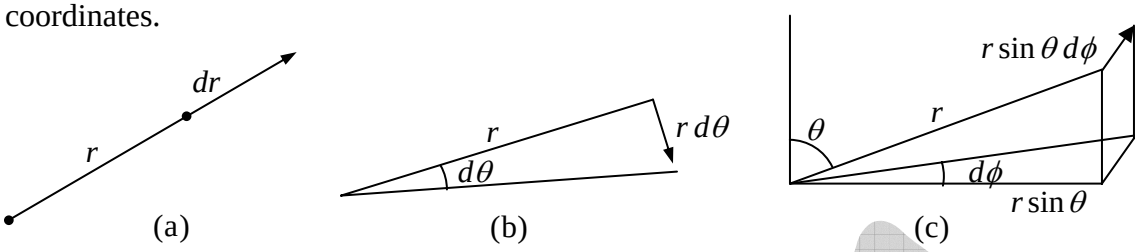
$$dl_\phi = r\sin\theta\,d\phi$$

Thus, the general infinitesimal displacement $d\vec{l}$ is

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\,d\phi\hat{\phi}$$

The range of *r* is $0 \to \infty$, θ goes from $0 \to \pi$, and ϕ goes from $0 \to 2\pi$.

This plays the role (in line integrals, for example) that $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ played in Cartesian coordinates.



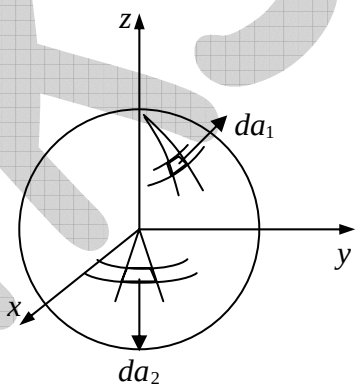
(a) (b) (c)

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r\, dl_\theta\, dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

If we are integrating over the surface of a sphere, for instance, then r is constant, whereas θ and ϕ change, so

$$d\vec{a}_1 = dl_\theta\, dl_\phi\, \hat{r} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$



on the other hand, if the surface lies in the xy plane, then θ is constant ($\theta = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_2 = dl_r\, dl_\phi\, \hat{\theta} = r\,dr\,d\phi\,\hat{\theta}$$

The vector derivatives in spherical coordinates:

Gradient: $$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

Divergence: $$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl: $$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{pmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{pmatrix}$$

Laplacian: $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 f}{\partial \phi^2}\right)$$

We can transform any vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ in Cartesian coordinates to Spherical polar coordinates ($\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$) using table given below:

where

$$A_r = \vec{A}.\hat{r} = A_x(\hat{x}.\hat{r}) + A_y(\hat{y}.\hat{r}) + A_z(\hat{z}.\hat{r})$$

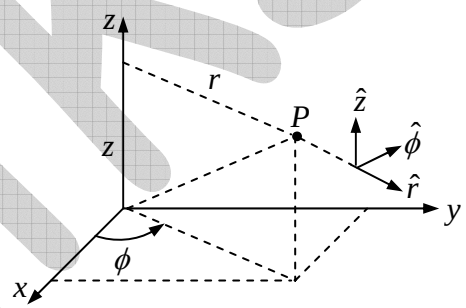
$$A_\theta = \vec{A}.\hat{\theta} = A_x(\hat{x}.\hat{\theta}) + A_y(\hat{y}.\hat{\theta}) + A_z(\hat{z}.\hat{\theta})$$

$$A_\phi = \vec{A}.\hat{\phi} = A_x(\hat{x}.\hat{\phi}) + A_y(\hat{y}.\hat{\phi}) + A_z(\hat{z}.\hat{\phi})$$

	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}.$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}.$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}.$	$\cos\theta$	$-\sin\theta$	0

1.4.2 Cylindrical Polar Coordinates

The cylindrical coordinates r, ϕ, z of a point P are defined in figure. Notice that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; r is the distance to P from the z axis, whereas the spherical coordinate r is the distance from the origin. The relation to Cartesian coordinates is



$$x = r\cos\phi, \quad y = r\sin\phi \ , \quad z = z$$

The infinitesimal displacements are

$$dl_r = dr, \quad dl_\phi = r d\phi, \quad dl_z = dz,$$

so $$d\vec{l} = dr\,\hat{r} + r\,d\phi\hat{\phi} + dz\,\hat{z}$$

and volume element is $d\tau = r\,dr\,d\phi\,dz$.

The range of r is $0 \to \infty$, ϕ goes from $0 \to 2\pi$, and z from $-\infty$ to ∞

The vector derivatives in cylindrical coordinates are:

Gradient: $$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

Divergence: $$\vec{\nabla}.\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Curl: $$\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}$$

Laplacian: $$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$$

We can transform any vector $\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$ in Cartesian coordinates to Cylindrical coordinates ($\vec{A}=A_r\hat{r}+A_\phi\hat{\phi}+A_z\hat{z}$) using table given below:

where

$$A_r=\vec{A}.\hat{r}=A_x(\hat{x}.\hat{r})+A_y(\hat{y}.\hat{r})+A_z(\hat{z}.\hat{r})$$

$$A_\phi=\vec{A}.\hat{\phi}=A_x(\hat{x}.\hat{\phi})+A_y(\hat{y}.\hat{\phi})+A_z(\hat{z}.\hat{\phi})$$

$$A_z=\vec{A}.\hat{z}=A_x(\hat{x}.\hat{z})+A_y(\hat{y}.\hat{z})+A_z(\hat{z}.\hat{z})$$

	$\hat{r}$	$\hat{\phi}$	$\hat{z}$
$\hat{x}.$	$\cos\phi$	$-\sin\phi$	0
$\hat{y}.$	$\sin\phi$	$\cos\phi$	0
$\hat{z}.$	0	0	1

Example: Find the gradient of a scalar function of position V where $V(x,y,z)=x^2y+e^z$. Calculate the magnitude of gradient at point $P(1,5,-2)$.

Solution: $V(x,y,z)=x^2y+e^z;\ \vec{\nabla}V=\frac{\partial V}{\partial x}\hat{x}+\frac{\partial V}{\partial y}\hat{y}+\frac{\partial V}{\partial z}\hat{z}=2xy\hat{x}+x^2\hat{y}+e^z\hat{z}$

At $P(1,5,-2)\Rightarrow\vec{\nabla}V=10\hat{x}+\hat{y}+0.1353\hat{z}\Rightarrow\left|\vec{\nabla}V\right|=\sqrt{10^2+1^2+0.1353^2}=10.056$

Example: In electrostatic field problems, the electric field is given by $\vec{E}=-\vec{\nabla}V$, where V is the scalar field potential. If $V=r^2\phi-2\theta$ in spherical coordinates, find $\vec{E}$.

Solution: $V=r^2\phi-2\theta$, In spherical coordinate, $\vec{\nabla}V=\frac{\partial V}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$

Substituting the suitable values, $\vec{\nabla}V=2r\phi\hat{r}-\frac{2}{r}\hat{\theta}+\frac{r^2}{r\sin\theta}\hat{\phi}$

$\Rightarrow\vec{E}=-\vec{\nabla}V=-2r\phi\hat{r}+\frac{2}{r}\hat{\theta}-\frac{r}{\sin\theta}\hat{\phi}$

Example: Given

(i) $\vec{A} = 2xy\hat{x} + z\hat{y} + yz^2\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$ at (2, -1, 3)

(ii) $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$

(iii) $\vec{A} = 10\hat{r} + 5\sin\theta\hat{\theta}$, Find $\vec{\nabla}\cdot\vec{A}$

Solution: (i) In Cartesian coordinates $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$A_x = 2xy, A_y = z, A_z = yz^2 \Rightarrow \vec{\nabla}\cdot\vec{A} = 2y + 0 + 2yz$, At (2, -1, 3), $\vec{\nabla}\cdot\vec{A} = -2 - 6 = -8$

(ii) In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$

$$A_r = 2r\cos^2\phi, A_\phi = 3r^2\sin z, A_z = 4z\sin^2\phi$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4\left(\cos^2\phi + \sin^2\phi\right) = 4$$

(iii) In spherical coordinates, $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$A_r = 10, A_\theta = 5\sin\theta, A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}20r + \frac{1}{r\sin\theta}10\sin\theta\cos\theta = (2 + \cos\theta)(10/r)$$

Example: Find the curl of the vector $\vec{A} = \left(e^{-r}/r\right)\hat{\theta}$

Solution: $\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

$$\vec{A} = \left(e^{-r}/r\right)\hat{\theta} \Rightarrow A_r = 0, A_\theta = \left(e^{-r}/r\right), A_\phi = 0$$

Substitute and solve the determinant $\vec{\nabla}\times\vec{A} = -\frac{e^{-r}}{r}\hat{\phi}$

Example: Find the nature of the following fields by determining divergence and curl.

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z}$

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ (Cylindrical coordinates)

Solution: (i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z}$

Divergence in Cartesian coordinates $\vec{\nabla}\cdot\vec{F}_1 = \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z} = 2x(1+5z)$

Divergence exists, so the field is non-solenoidal.

$$\vec{\nabla}\times\vec{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix} = -5z^2\hat{y} + 2y\hat{z}$$

The field has a curl so it is rotational.

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ in cylindrical coordinates.

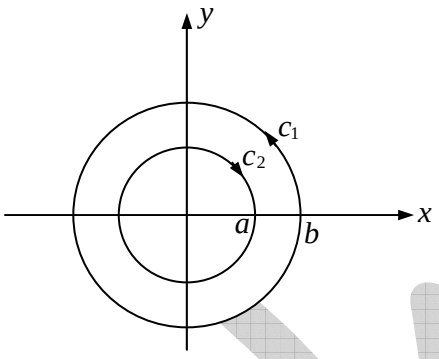
In cylindrical coordinates, Divergence $\vec{\nabla}\cdot\vec{F}_2 = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_{2r}\right) + \frac{1}{r}\frac{\partial F_{2\phi}}{\partial \phi} + \frac{\partial F_{2z}}{\partial z} = \frac{-150}{r^3}$

The field is non-solenoid.

$$\vec{\nabla}\times\vec{F}_2 = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \left(\frac{150}{r^2}\right) & 10r & 0 \end{vmatrix} = \frac{10}{r}\hat{z}$$

The vector field $\vec{F}_2$ has non-zero curl so it is rotational.

Example: Given $\vec{A} = 2r\cos\phi\hat{r} + r\hat{\phi}$ in cylindrical coordinates. Find $\oint_{C_1}\vec{A}\cdot d\vec{l} + \oint_{C_2}\vec{A}\cdot d\vec{l}$ where c_1 and c_2 are contours shown in figure.



Solution: In cylindrical coordinate system $d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$, $\vec{A} = 2r\cos\phi\hat{r} + r\hat{\phi}$

$$\vec{A}\cdot d\vec{l} = 2r\cos\phi dr + r^2 d\phi$$

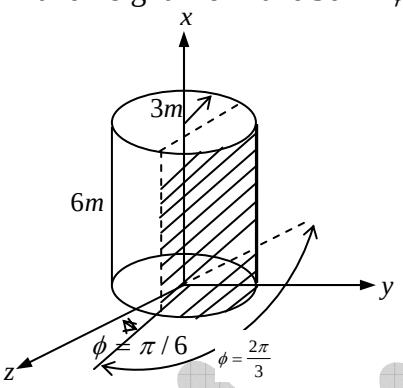
In figure on curve c_1, ϕ varies from 0 to $2\pi, r = b$ and $dr = 0$

$$\oint_{c_1}\vec{A}\cdot d\vec{l} = \int_{\substack{\phi=0\\ r=b}}^{2\pi} r^2 d\phi = 2\pi b^2$$

On curve $c_2, r = a, \phi$ varies from 0 to -2π, and $dr = 0 \Rightarrow \oint_{c_2}\vec{A}\cdot d\vec{l} = \int_{\substack{\phi=0\\ r=a}}^{-2\pi} r^2 d\phi = -2\pi a^2$

So, $\oint_{c_1}\vec{A}\cdot d\vec{l} + \oint_{c_2}\vec{A}\cdot d\vec{l} = 2\pi\left(b^2 - a^2\right)$

Example: Use the cylindrical coordinate system to find the area of a curved surface on the right circular cylinder having radius = 3 *m* and height = 6 *m* and $30^0 \le \phi \le 120^0$.



Solution: From figure, surface area is required for a cylinder when *r* = 3m, *z* = 0 to 6m,

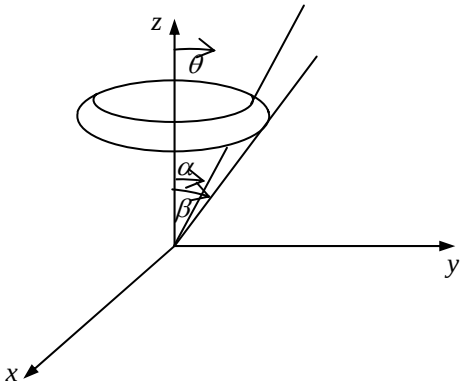
$$30^0 \le \phi \le 120^0 \text{ or } \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$$

In cylindrical coordinate system, the elemental surface area as scalar is $d\vec{a} = r\,d\phi\,dz\hat{r}$

Taking the magnitude only

$$A = \int_S da = \int_{\phi=\pi/6}^{2\pi/3} \int_{z=0}^{6} r\,d\phi\,dz = 3\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)6 = 9\pi\, m^2$$

Example: Use spherical coordinate system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius '*a*'. Calculate the area when $\alpha = 0$ and $\beta = \pi$.



Solution: Sphere has radius '*a*' and θ varies between α and β.

For fixed radius the elemental surface is

$da = (r\sin\theta\,d\phi)(r\,d\theta) = r^2\sin\theta\,d\theta\,d\phi$

Area

$$A = \int_{\theta=\alpha}^{\beta}\int_{\phi=0}^{2\pi} r^2\sin\theta\,d\theta\,d\phi = 2\pi a^2 \int_{\theta=\alpha}^{\beta}\sin\theta\,d\theta$$

$$= 2\pi a^2(\cos\alpha - \cos\beta)$$

For $\alpha = 0, \beta = \pi$, Area $= 2\pi a^2(1+1) = 4\pi a^2$, is surface area of the sphere.

Example: A vector field $\vec{A} = \left(\frac{5r^2}{4}\right)\hat{r}$ is given in spherical coordinates. Evaluate both sides of Divergence Theorem for the volume enclosed between

(i) $r = 1$ and $r = 2$, and

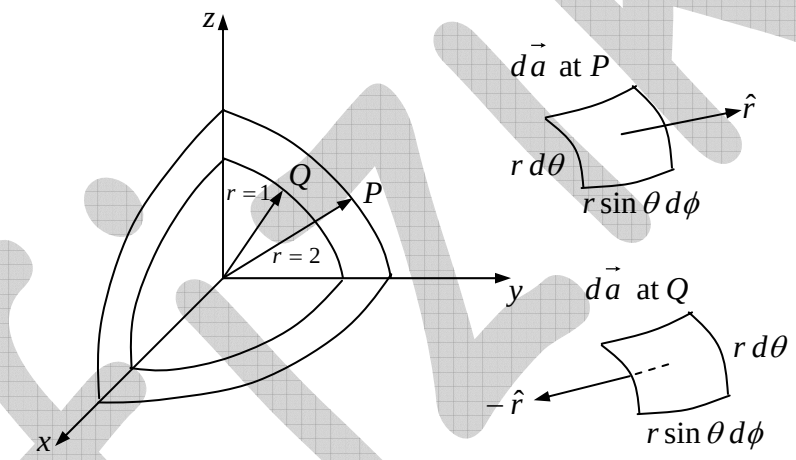
(ii) $\theta = 0$ to $\theta = \frac{\pi}{4}$ and $r = 4$.

Solution: Divergence theorem states that $\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oint_S \vec{A}\cdot d\vec{a}$

Since $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$A_r = \frac{5r^2}{4}, \quad A_\theta = 0, \quad A_\phi = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{5}{4}r^2\right) = 5r$

(i)



$$L.H.S = \int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \int_v (5r)r^2\sin\theta\,dr\,d\theta\,d\phi = \int_{r=1}^{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 5r^3\sin\theta\,dr\,d\theta\,d\phi = 75\pi$$

$$\text{R.H.S } \oint_S \vec{A}\cdot d\vec{a} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(r^2\sin\theta\,d\theta\,d\phi\hat{r}\right) + \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(-r^2\sin\theta\,d\theta\,d\phi\hat{r}\right)$$

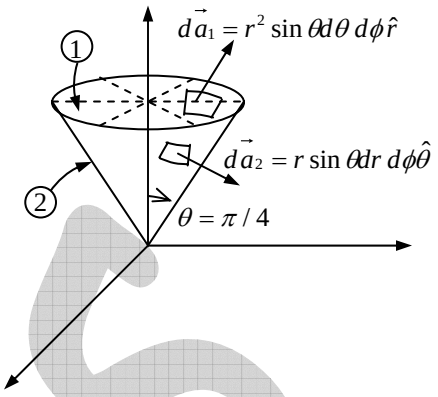
$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{5}{4}(2)^4\sin\theta\,d\theta\,d\phi - \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{5}{4}(1)^4\sin\theta\,d\theta\,d\phi = 75\pi$$

So L.H.S. = R.H.S. = 75π

Divergence theorem proved.

(ii) L.H.S. of Divergence Theorem

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \int_{r=0}^{4}\int_{\theta=0}^{\pi/4}\int_{\phi=0}^{2\pi}(5r)\cdot r^2\sin\theta\, dr\, d\theta\, d\phi = 588.91$$



R.H.S. of Divergence Theorem

$$\oint_S \vec{A}\cdot d\vec{a} = \int_{S_1}\vec{A}\cdot d\vec{a} + \int_{S_2}\vec{A}\cdot d\vec{a}$$

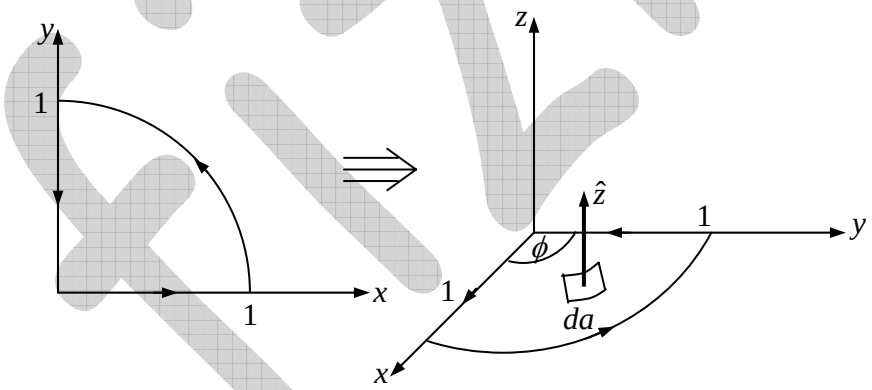
$$= \int_{S_1}\left(\frac{5}{4}r^2\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\hat{r}\right) + \int_{S_2}\left(\frac{5}{4}r^2\hat{r}\right)\cdot\left(r\sin\theta\, dr\, d\phi\hat{\theta}\right)$$

$$= \int_{\theta=0}^{\pi/4}\int_{\phi=0}^{2\pi}\frac{5}{4}(4)^4\sin\theta\, d\theta\, d\phi + 0 = 588.91$$

L.H.S. = R.H.S. = 588.91

Divergence theorem proved.

Example: Given $\vec{A} = 2r\cos\phi\hat{r} + r\hat{\phi}$ in cylindrical coordinates. For the contour shown in figure, verify the Stokes' Theorem.



Solution: Stokes' Theorem $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint\vec{A}\cdot d\vec{l}$

In cylindrical coordinates, $\vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}$

$A_r = 2r\cos\phi,\quad A_\phi = r\,,A_z = 0$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{-\partial}{\partial z}\left(r^2\right)\right]\hat{r} + \left[\frac{\partial}{\partial z}\left(2r\cos\phi\right)\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r^2\right)\frac{\partial}{\partial\phi}\left(2r\cos\phi\right)\right]\hat{z} = \left(2+2\sin\phi\right)\hat{z}$$

$$\text{L.H.S.} = \int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\left(2+2\sin\phi\right)\hat{z}\cdot\left(r\,dr\,d\phi\hat{z}\right) = \frac{\pi}{2}+1$$

$$\text{R.H.S.} = \oint \vec{A}\cdot d\vec{l} = \int_{r=0,1}\vec{A}\cdot d\vec{l} + \int_{\phi=0,\pi/2}\vec{A}\cdot d\vec{l} + \int_{r=1,0}\vec{A}\cdot d\vec{l}$$

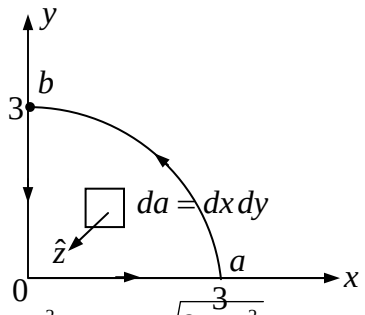
$$\vec{A}\cdot d\vec{l} = \left(2r\cos\phi\hat{r} + r\hat{\phi}\right)\cdot\left(dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}\right) = 2r\cos\phi\,dr + r^2 d\phi$$

$$\oint \vec{A}\cdot d\vec{l} = \int_{\substack{r=0\\ at\ \phi=0}}^{1} 2r\cos\phi\,dr + \int_{\substack{\phi=0\\ at\ r=1}}^{\pi/2} r^2 d\phi + \int_{\substack{r=1\\ at\ \phi=\pi/2}}^{0} 2r\cos\phi\,dr \quad = 1+\frac{\pi}{2}+0 = 1+\frac{\pi}{2}$$

$$\text{L.H.S.} = \text{R.H.S.} = 1+\frac{\pi}{2}$$

Example: Given a vector field $\vec{A} = xy\hat{x} - 2x\hat{y}$. Verify stokes,' theorem over the path shown in figure.

Solution: Stokes' theorem $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$



$$\vec{\nabla}\times\vec{A} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & -2x & 0\end{vmatrix} = -\left(2+x\right)\hat{z}$$

$$\text{L.H.S.} = \int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}}\left(\vec{\nabla}\times\vec{A}\right)\cdot\left(dx\,dy\hat{z}\right),\ \text{since}\ r^2 = x^2+y^2\ \text{or}\ x = \sqrt{9-y^2}$$

$$= \int_0^3\left[\int_0^{\sqrt{9-y^2}} -\left(2+x\right)dx\,dy\right] = \int_0^3\left[-2x - \frac{x^2}{2}\Big|_0^{\sqrt{9-y^2}}\,dy\right] = \int_0^3 -\left[+2\sqrt{9-y^2} + \left(\frac{9-y^2}{2}\right)\right]dy$$

$$= -\left[y\sqrt{9-y^2} + 9\sin^{-1}\frac{y}{3} + \frac{9}{2}y - \frac{y^3}{6}\right]_0^3 = -9\left(1+\frac{\pi}{2}\right)$$

$$\text{R.H.S.} = \oint \vec{A}\cdot d\vec{l} = \int_{0,a}\vec{A}\cdot d\vec{l} + \int_{a,b}\vec{A}\cdot d\vec{l} + \int_{b,0}\vec{A}\cdot d\vec{l}$$

On 0a, $y = 0$; $\int \vec{A} \cdot d\vec{l} = \int -2x\,dy = 0$

On ab; $\int \vec{A} \cdot d\vec{l} = \int (xy\,dx - 2x\,dy) = \int_3^0 x\sqrt{9-x^2}\,dx - 2\int_0^3 \sqrt{9-y^2}\,dy$

(Equation of quarter circle $x^2 + y^2 = 9$; $0 \le x, y \le 3$)

$$\int \vec{A} \cdot d\vec{l} = -\frac{1}{3}\left(9-x^2\right)^{3/2}\Big|_3^0 - \left[y\sqrt{9-y^2} + 9\sin^{-1}\frac{y}{3}\right]_0^3 = -9\left(1+\frac{\pi}{2}\right)$$

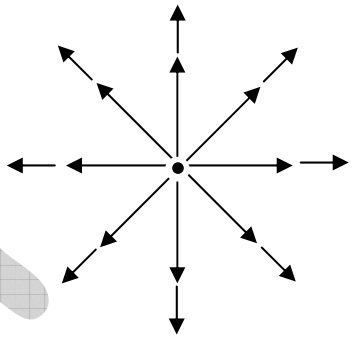
On b0, x = 0; $\int \vec{A} \cdot d\vec{l} = 0$

$$\Rightarrow \oint \vec{A} \cdot d\vec{l} = -9\left(1+\frac{\pi}{2}\right)$$

L.H.S. = R.H.S $= -9\left(1+\frac{\pi}{2}\right)$

1.5 The Dirac Delta Function

1.5.1 The Divergence of $\hat{r}/r^2$



Consider the vector function $\vec{A} = \frac{1}{r^2}\hat{r}$

At every location, $\vec{A}$ is directed radially outward. When we calculate the divergence we get precisely zero:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

The plot thickens if you apply the divergence theorem to this function. Suppose we integrate over a sphere of radius *R*, centered at the origin; the surface integral is

$$\oint \vec{A}\cdot d\vec{a} = \int\left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\,\hat{r}\right) = \left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

But the volume integral, $\int\left(\vec{\nabla}\cdot\vec{A}\right)d\tau$, is zero. Does this mean that the divergence theorem is false?

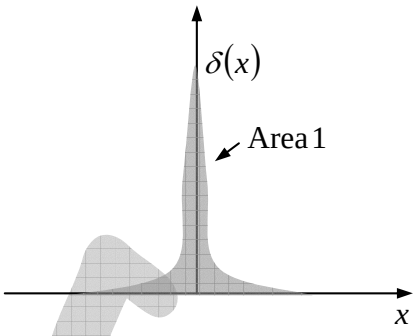
The source of the problem is the point $r = 0$, where $\vec{A}$ blows up. It is quite true that $\vec{\nabla}\cdot\vec{A} = 0$ everywhere except the origin, but right at the origin the situation is more complicated.

Notice that the surface integral is independent of *R*; if the divergence theorem is right (and it is), we should get $\int\left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 4\pi$ for any sphere centered at the origin, no matter how small. Evidently the entire contribution must be coming from the point $r = 0$!

Thus, $\vec{\nabla}\cdot\vec{A}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral (over any volume containing that point) is 4π. No ordinary function behaves like that. (On the other hand, a physical example does come to mind: the density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its integral is finite namely, the mass of the particle.) What we have stumbled on is a mathematical object known to physicists as the ***Dirac delta function***.

1.5.2 The One- Dimensional Dirac Delta Function

The one dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1 (as shown in figure).



That is to say:

$$\delta(x)=\begin{cases}0 & if\ x\neq 0\\ \infty, & if\ x=0\end{cases}$$

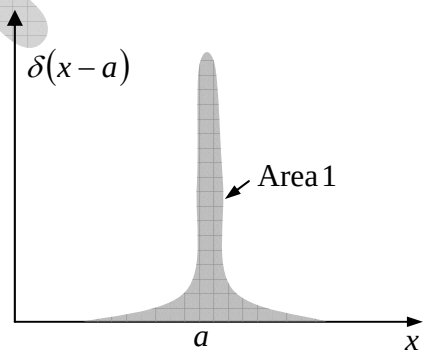
and $$\int_{-\infty}^{\infty}\delta(x)dx=1$$

If $f(x)$ is some "ordinary" function then the product $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows that

$$f(x)\delta(x)=f(0)\delta(x)$$

Of course, we can shift the spike from $x = 0$ to some other point, $x = a$:

$$\delta(x-a)=\begin{cases}0, & if\ x\neq a\\ \infty, & if\ x=a\end{cases}\text{ with }\int_{-\infty}^{\infty}\delta(x-a)dx=1$$



also

$$f(x)\delta(x-a)=f(a)\delta(x-a)$$

and

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$$

1.5.3 The Three-Dimensional Delta Function

It is an easy matter to generalize the delta function to three dimensions:

$$\delta^3(\vec{r})=\delta(x)\delta(y)\delta(z)$$

(As always, $\vec{r}=x\,\hat{x}+y\,\hat{y}+z\,\hat{z}$ is the position vector, extending from the origin to the point (x, y, z)). This three-dimensional delta function is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1

$$\int_{all\ space} \delta^3(\vec{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz = 1$$

and

$$\int_{all\ space} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(\vec{a})$$

Since the divergence of $\hat{r}/r^2$ is zero everywhere except at the origin, and yet its integral over any volume containing the origin is a constant (4π). These are precisely the defining conditions for the Dirac delta function; evidently

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

1.6 The Theory of Vector Fields

1.6.1 The Helmholtz Theorem

1.6.2 Potentials

If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be written as the gradient of a scalar potential (V):

$$\vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \vec{F} = -\vec{\nabla} V$$

(The minus sign is purely conventional.)

Theorem 1: Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, $\vec{F}$ satisfies one if and only if it satisfies all the others):

(a) $\vec{\nabla} \times \vec{F} = 0$ everywhere.

(b) $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.

(c) $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.

(d) $\vec{F}$ is the gradient of some scalar, $\vec{F} = -\vec{\nabla} V$.

The scalar potential is not unique-any constant can be added to V with impunity, since this will not affect its gradient.

If the divergence of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be expressed as the curl of a vector potential $\left(\vec{A}\right)$:

$$\vec{\nabla} \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \vec{\nabla} \times \vec{A}$$

That's the main conclusion of the following theorem:

Theorem 2: Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

(a) $\vec{\nabla} \cdot \vec{F} = 0$ everywhere.

(b) $\int \vec{F} \cdot d\vec{a}$ is independent of surface, for any given boundary line.

(c) $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface.

(d) $\vec{F}$ is the curl of some vector, $\vec{F} = \vec{\nabla} \times \vec{A}$.

The vector potential is not unique-the gradient of any scalar function can be added to $\vec{A}$ without affecting the curl, since the curl of a gradient is zero.

MCQ (Multiple Choice Questions)

Q1. Let $\vec{a}$ and $\vec{b}$ be two distinct three dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

(a) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ (b) $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ (c) $\frac{(\vec{a}\cdot\vec{b})b}{b^2}$ (d) $\frac{(\vec{b}\cdot\vec{a})\vec{a}}{a^2}$

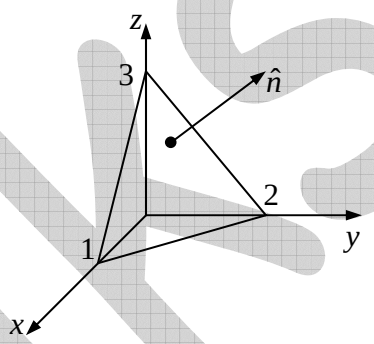
Q2. The components of the unit vector $\hat{n}$ perpendicular to the plane shown in the figure given below is:

(a) $\hat{n}=\frac{6\hat{x}+3\hat{y}+2\hat{z}}{7}$

(b) $\hat{n}=\frac{3\hat{x}+6\hat{y}+2\hat{z}}{7}$

(c) $\hat{n}=\frac{6\hat{x}+2\hat{y}+3\hat{z}}{7}$

(d) $\hat{n}=\frac{2\hat{x}+3\hat{y}+6\hat{z}}{7}$



Q3. The equation of the plane that is tangent to the surface $xyz=8$ at the point $(1,2,4)$ is

(a) $x+2y+4z=12$ (b) $4x+2y+z=12$

(c) $x+4y+2=0$ (d) $x+y+z=7$

Q4. A vector perpendicular to any vector that lies on the plane defined by $x+y+z=5$, is

(a) $\hat{i}+\hat{j}$ (b) $\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}+\hat{k}$ (d) $2\hat{i}+3\hat{j}+5\hat{k}$

Q5. The unit normal vector of the point $\left[\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right]$ on the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

(a) $\frac{bc\hat{i}+ca\hat{j}+ab\hat{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$ (b) $\frac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

(c) $\frac{b\hat{i}+c\hat{j}+a\hat{k}}{\sqrt{a^2+b^2+c^2}}$ (d) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

Q6. The equation of a surface of revolution is $z = \pm\sqrt{\frac{3}{2}x^2 + \frac{3}{2}y^2}$. The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}},0,1\right)$ is

(a) $\sqrt{\frac{3}{5}}\hat{i} + \frac{2}{\sqrt{10}}\hat{k}$ (b) $\sqrt{\frac{3}{5}}\hat{i} - \frac{2}{\sqrt{10}}\hat{k}$

(c) $\sqrt{\frac{3}{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}$ (d) $\sqrt{\frac{3}{10}}\hat{i} + \frac{2}{\sqrt{10}}\hat{k}$

Q7. Let $\vec{r}$ denote the position vector of any point in three-dimensional space, and $r = |\vec{r}|$. Then

(a) $\vec{\nabla}\cdot\vec{r} = 0$ and $\vec{\nabla}\times\vec{r} = \vec{r}/r$ (b) $\vec{\nabla}\cdot\vec{r} = 0$ and $\nabla^2 r = 0$

(c) $\vec{\nabla}\cdot\vec{r} = 3$ and $\nabla^2\vec{r} = \vec{r}/r^2$ (d) $\vec{\nabla}\cdot\vec{r} = 3$ and $\vec{\nabla}\times\vec{r} = 0$

Q8. For vector function $\vec{A} = 10\hat{r} + 5\sin\theta\hat{\theta}$ (in spherical polar coordinate) the value of $\vec{\nabla}.\vec{A}$ is:

(a) $(1+\sin\theta)(10/r)$ (b) $(1+\cos\theta)(10/r)$

(c) $(2+\sin\theta)(10/r)$ (d) $(2+\cos\theta)(10/r)$

Q9. For vector function $\vec{F} = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ (in cylindrical coordinate) then $\vec{\nabla}.\vec{F}$ is

(a) $\frac{-150}{r^3}$ (b) $\frac{150}{r^3}$ (c) $\frac{-150}{r^2}$ (d) $\frac{150}{r^2}$

Q10. For vector function $\vec{A} = \left(e^{-r}/r\right)\hat{\theta}$ (in spherical polar coordinate) then $\vec{\nabla}\times\vec{A}$ is

(a) $\frac{e^{-r}}{r}\hat{\phi}$ (b) $-\frac{e^{-r}}{r}\hat{\phi}$ (c) $\frac{e^{-r}}{r}\hat{\theta}$ (d) $-\frac{e^{-r}}{r}\hat{\theta}$

Q11. A vector $\vec{A} = k\hat{\phi}$ is given (where k is a constant), in cylindrical coordinates. Then the value of $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$ is:

(a) $\frac{k}{r}\hat{z}$ (b) $\frac{k}{r}\hat{\phi}$ (c) $\frac{k}{r^2}\hat{z}$ (d) $\frac{k}{r^2}\hat{\phi}$

Q12. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla}\times\vec{F}=0$ (b) $\vec{\nabla}\cdot\vec{F}=0$ (c) $\vec{\nabla}V=0$ (d) $\nabla^2 V=0$

Q13. If $\vec{F}$ is a constant vector and $\vec{r}$ is the position vector then $\vec{\nabla}\left(\vec{F}\cdot\vec{r}\right)$ would be

(a) $\left(\vec{\nabla}\cdot\vec{r}\right)\vec{F}$ (b) $\vec{F}$ (c) $\left(\vec{\nabla}\cdot\vec{F}\right)\vec{r}$ (d) $|\vec{r}|\vec{F}$

Q14. If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}\left(\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right)$ is

(a) $\vec{A}\cdot\vec{B}$ (b) $\vec{A}\times\vec{B}$ (c) $\vec{r}$ (d) zero

Q15. If $\vec{A}=\hat{i}yz+\hat{j}xz+\hat{k}xy$, then the integral $\oint_C \vec{A}\cdot d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by $x=0, x=a$ and $y=0,\ y=b$) is

(a) $\frac{1}{2}\left(a^3+b^3\right)$ (b) $\pi\left(ab^2+a^2b\right)$ (c) $\pi\left(a^3+b^3\right)$ (d) 0

Q16. If the surface integral of the field $\vec{A}(x,y,z)=2kx\,\hat{i}+ly\,\hat{j}-3mz\,\hat{k}$ over the closed surface of an arbitrary unit sphere is to be zero, then the relationship between k, l and m is

(a) $6k+l-6m=0$ (b) $2k+l-3m=0$

(c) $3k+6l-2m=0$ (d) $2/k+1/l-3/m=0$

Q17. For the vector field $\vec{A}=xz^2\hat{i}-yz^2\,\hat{j}+z\left(x^2-y^2\right)\hat{k}$, the volume integral of the divergence of $\vec{A}$ out of the region defined by $-a\le x\le a$, $-b\le y\le b$ and $0\le z\le c$ is:

(a) $\frac{4}{3}abc\left[a^2-b^2\right]$ (b) $\frac{2}{3}abc\left[a^2-b^2\right]$

(c) $\frac{1}{3}abc\left[a^2-b^2\right]$ (d) $abc\left[a^2-b^2\right]$

Q18. Consider a constant vector field $\vec{v}=v_0\hat{k}$. If $\vec{v}=\vec{\nabla}\times\vec{u}$ then one of the many possible vectors $\vec{u}$ is

(a) $v_0x\hat{i}$ (b) $v_0x\,\hat{j}$ (c) $v_0\,\hat{i}$ (d) $v_0\,\hat{j}$

Q19. Consider a vector field $\vec{v} = v_0\hat{k}$ and $\vec{u} = v_0 x\,\hat{j}$ where $\vec{v} = \vec{\nabla}\times\vec{u}$. Then the flux associated with the field $\vec{v}$ through the curved hemispherical surface defined by $x^2 + y^2 + z^2 = r^2;\ z > 0$ is

(a) 0 (b) $\pi v_0 r^2$ (c) $2\pi v_0 r^2$ (d) $3\pi v_0 r^2$

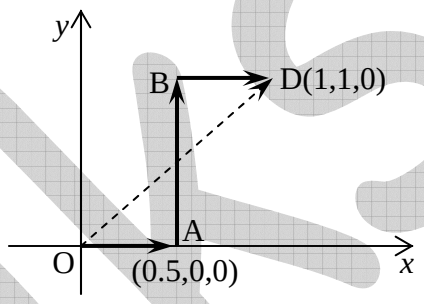
Q20. Consider a vector force $\vec{F}(x,y) = k\left[\left(x^2 + y^2\right)\hat{i} + 2xy\ \hat{j}\right]$. Here $1 Fm^{-2}$. The work done by this force in moving a particle from the origin $O(0,0,0)$ to the point $D(1,1,0)$ on the $z = 0$ plane along the path $OABD$ as shown in the figure is: (where the coordinates are measured in meters)

(a) $\frac{1}{3}$

(b) $\frac{2}{3}$

(c) $\frac{4}{3}$

(d) 0



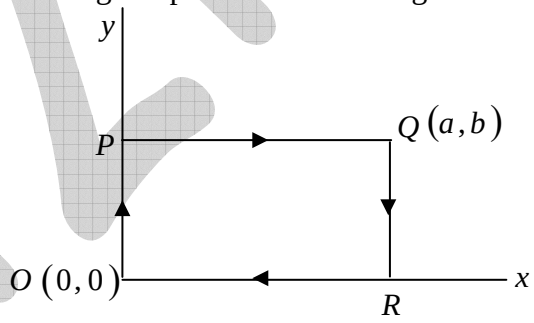
Q21. Consider force field $\vec{F}(x,y) = \left(x^2 - y^2\right)\hat{i} + 2xy\,\hat{j}$. Then the work done when an object moves from $O \to P \to Q \to R \to O$ along the rectangular path as shown in figure is:

(a) $-2ab^2$

(b) $+2ab^2$

(c) $-ab^2$

(d) $+ab^2$



Q22. Which of the following is correct expression for $\delta(kx)$, where k is any (nonzero constant) (In particular $\delta(-x) = \delta(x)$).

(a) $\delta(kx) = \frac{1}{|k|}\delta(x)$ (b) $\delta(kx) = |k|\,\delta(x)$

(c) $\delta(kx) = -\frac{1}{|k|}\delta(x)$ (d) $\delta(kx) = -|k|\,\delta(x)$

Q. 23. Evaluate the integral $\int_0^1 x^3\delta(x-2)\,dx$

(a) 0 (b) 4 (c) 8 (d) 12

Q24. Evaluate the integral $J=\int_v (r^2+2)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau$ where v is a sphere of radius R centered at the origin.

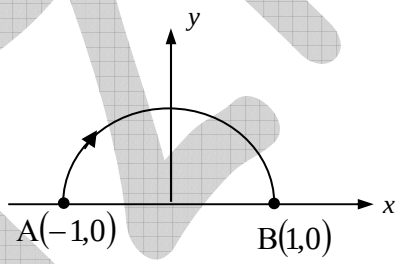
(a) 2π (b) 4π (c) 6π (d) 8π

Q25. Evaluate the integral $\int_0^3 x^3\delta(x-2)dx$

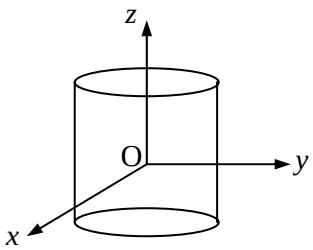
(a) 0 (b) 4 (c) 8 (d) 12

NAT (Numerical Answer Type)

Q26. The line integral $\int_A^B \vec{F}\cdot d\vec{l}$, where $\vec{F}=\frac{x}{\sqrt{x^2+y^2}}\hat{x}+\frac{y}{\sqrt{x^2+y^2}}\hat{y}$, along the semi-circular path as shown in the figure below is.....



Q27. Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r}=\hat{i}x+\hat{j}y+\hat{k}z$ be the position vector and $\hat{n}$ a unit vector normal to the surface. The surface integral $\int_S \vec{r}\cdot\hat{n}\,ds$ over the closed surface of the cylinder is $\alpha\pi a^2 h$. Then the value of α is.....



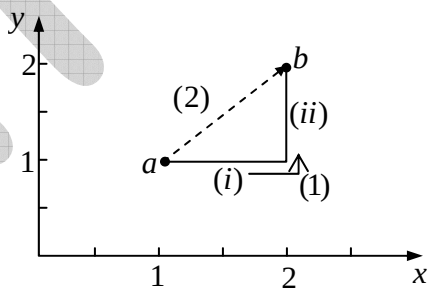
Q28. For vector function $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$ the value of $\vec{\nabla}.\vec{A}$ is.....

Q29. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120^o with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60^o if $b = \alpha a$. Then the value of α is.....

Q30. If S is the closed surface enclosing a volume V and $\hat{n}$ is the unit normal vector to the surface and $\vec{r}$ is the position vector, then the value of the following integral $\int\int_S \vec{r}.\hat{n}dS$ is αV. Then the value of α is ……

MSQ (Multiple Select Questions)

Q31. A vector function $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ is given and two specified paths from a *to* b are shown in the figure given below. Coordinates of point a is $(1,1,0)$ and that of point b is $(2,2,0)$. Then which of the following statements are



(a) $\int_a^b \vec{A}\cdot d\vec{l} = 11$ along path 1.

(b) $\int_a^b \vec{A}\cdot d\vec{l} = 10$ along path 2.

(c) $\oint \vec{A}\cdot d\vec{l} = -1$ for the loop that goes from a to b along (1) and returns to a along (2).

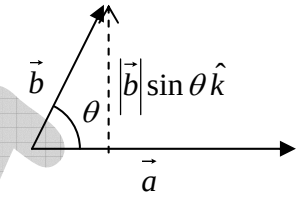
(d) $\oint \vec{A}\cdot d\vec{l} = 1$ for the loop that goes from a to b along (1) and returns to a along (2).

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (a)

Solution: $\vec{a}\times\vec{b} = ab\sin\theta\,\hat{n}$ where $\hat{n}$ is perpendicular to plane containing $\vec{a}$ and $\vec{b}$ and pointing upwards.



$$\vec{a}\times\left(\vec{a}\times\vec{b}\right) = ab\sin\theta\left(\vec{a}\times\hat{n}\right) = -a^2 b\sin\theta\,\hat{k}$$

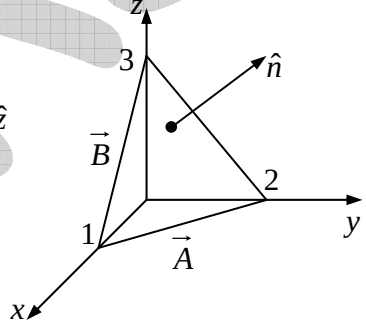
$$b\sin\theta\,\hat{k} = \frac{-\vec{a}\times\left(\vec{a}\times\vec{b}\right)}{a^2} \Rightarrow b\sin\theta\,\hat{k} = \frac{\vec{a}\times\left(\vec{b}\times\vec{a}\right)}{a^2}.$$

Ans. 2: (a)

Solution: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

$$\vec{A} = -\hat{x} + 2\hat{y}; \quad \vec{B} = -\hat{x} + 3\hat{z}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \hat{x}(6-0) + \hat{y}(-3+0) + \hat{z}(0+2) = 6\hat{x} + 3\hat{y} + 2\hat{z}$$



$$\Rightarrow \hat{n} = \frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{7}$$

Ans. 3: (b)

Solution: To get a normal at the surface let's take the gradient

$$\vec{\nabla}(xyz) = yz\hat{i} + zx\hat{j} + \hat{k}xy = 8\hat{i} + 4\hat{j} + 2\hat{k}$$

We want a plane perpendicular to this so: $\left(\vec{r} - \vec{r_0}\right)\cdot\frac{\left(8\hat{i} + 4\hat{j} + 2\hat{k}\right)}{\sqrt{64+16+4}} = 0.$

$$\left[(x-1)\hat{i} + (y-2)\hat{j} + (z-4)\hat{k}\right]\cdot\left[8\hat{i} + 4\hat{j} + 2\hat{k}\right] = 0 \Rightarrow 4x + 2y + z = 12.$$

Ans. 4: (c)

Solution: Let $\phi = x + y + z - 5 \Rightarrow \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z - 5) = \hat{i} + \hat{j} + \hat{k}$

Ans. 5: (a)

Solution: Here $\phi = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$. Unit normal vector is $\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$.

So, $\vec{\nabla}\phi = \left(i\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) = \frac{2x\hat{i}}{a^2}+\frac{2y\hat{j}}{b^2}+\frac{2z\hat{k}}{c^2}$

$$\vec{\nabla}\phi\Big|_{\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)} = \frac{2}{a\sqrt{3}}\hat{i}+\frac{2}{b\sqrt{3}}\hat{j}+\frac{2}{c\sqrt{3}}\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{\frac{4}{3a^2}+\frac{4}{3b^2}+\frac{4}{3c^2}} = \frac{2}{\sqrt{3}}\sqrt{\frac{b^2c^2+a^2c^2+a^2c^2}{a^2b^2c^2}}$$

$$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\Bigg|_{\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2}{a\sqrt{3}}\hat{i}+\frac{2}{b\sqrt{3}}\hat{j}+\frac{2}{c\sqrt{3}}\hat{k}}{\frac{2}{\sqrt{3}}\frac{\sqrt{b^2c^2+c^2a^2+a^2b^2}}{abc}} = \frac{bc\hat{i}+ca\hat{j}+ab\hat{k}}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$$

Ans. 6: (b)

Solution: $z = \pm\sqrt{\frac{3}{2}x^2+\frac{3}{2}y^2} \Rightarrow z^2 = \frac{3}{2}x^2+\frac{3}{2}y^2 \Rightarrow 3x^2+3y^2-2z^2=0$

Let $V = 3x^2+3y^2-2z^2$, Taking gradient $\Rightarrow \vec{\nabla}V = 6x\hat{x}+6y\hat{y}-4z\hat{z}$.

The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}},0,1\right)$ is $\hat{n} = \frac{\vec{\nabla}V}{|\vec{\nabla}V|}$. Thus

$$\hat{n} = \frac{6\sqrt{\frac{2}{3}}\hat{x}+6\times0\hat{y}-4\times1\hat{z}}{\sqrt{36\times\frac{2}{3}+16}} = \frac{6\sqrt{\frac{2}{3}}\hat{x}-4\hat{z}}{\sqrt{40}} = \sqrt{\frac{3}{5}}\hat{x}-\frac{2}{\sqrt{10}}\hat{z}$$

Ans. 7: (d)

Solution: $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\vec{\nabla}\times\vec{r}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix}=\hat{x}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)+\hat{y}\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)+\hat{z}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)=0$$

Ans. 8: (d)

Solution: $\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$\because A_r=10,\ \ A_\theta=5\sin\theta,\ \ A_\phi=0$$

$$\Rightarrow\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times10\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\times5\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial(0)}{\partial\phi}$$

$$\Rightarrow\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2}20r+\frac{1}{r\sin\theta}10\sin\theta\cos\theta=\left(2+\cos\theta\right)(10/r)$$

Ans. 9: (a)

Solution: $\vec{\nabla}\cdot\vec{F}=\frac{1}{r}\frac{\partial}{\partial r}\left(rF_r\right)+\frac{1}{r}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}\Rightarrow\vec{\nabla}\cdot\vec{F}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{150}{r^2}\right)+\frac{1}{r}\frac{\partial(10)}{\partial\phi}+\frac{\partial(0)}{\partial z}=\frac{-150}{r^3}$

Ans. 10: (b)

Solution: $\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$

$$\Rightarrow\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & e^{-r} & 0\end{vmatrix}\qquad\because A_r=0,\ A_\theta=\left(e^{-r}/r\right),\ A_\phi=0$$

$$\Rightarrow\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\times r\sin\theta\hat{\phi}\left(\frac{\partial e^{-r}}{\partial r}\right)=-\frac{e^{-r}}{r}\hat{\phi}$$

Ans. 11: (d)

Solution: $\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{pmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ 0 & rk & 0\end{pmatrix}=\frac{k}{r}\hat{z}\Rightarrow\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\frac{1}{r}\begin{pmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ 0 & 0 & k/r\end{pmatrix}=-\frac{r\hat{\phi}}{r}\left(-\frac{k}{r^2}\right)=\frac{k}{r^2}\hat{\phi}$

Ans. 12: (a)

Solution: Since $\vec{F}$ is derivative from potential $V(r)$ and $\vec{F}=-\vec{\nabla}V(r)$

$\Rightarrow\vec{\nabla}\times\vec{F}=-\vec{\nabla}\times\left(\vec{\nabla}V\right)=0$.

Ans. 13: (b)

Solution: Let $\vec{F}=F_0\left(\hat{x}+\hat{y}+\hat{z}\right)$ and $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}\Rightarrow\vec{F}.\vec{r}=F_0\left(x+y+z\right)$.

Thus $\vec{\nabla}\left(\vec{F}\cdot\vec{r}\right)=F_0\left(\hat{x}+\hat{y}+\hat{z}\right)=\vec{F}$

Ans. 14: (d)

Solution: Let $\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$, $\vec{B}=B_x\hat{x}+B_y\hat{y}+B_z\hat{z}$ and $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$.

$\vec{B}\times\vec{r}=\left(B_yz-yB_z\right)\hat{x}+\left(B_zx-zB_x\right)\hat{y}+\left(B_xy-xB_y\right)\hat{z}$

$\Rightarrow\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=A_x\left(B_yz-yB_z\right)+A_y\left(B_zx-zB_x\right)+A_z\left(B_xy-xB_y\right)$

$\Rightarrow\vec{\nabla}\left[\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right]=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}=\vec{A}\times\vec{B}$

Ans. 15: (d)

Solution: $\oint_C\vec{A}\cdot d\vec{l}=\int_S\left(\vec{\nabla}\times\vec{A}\right)d\vec{a}=0\quad\because\vec{\nabla}\times\vec{A}=0$

Ans. 16: (b)

Solution: It is given that $\oint_S\vec{A}.d\vec{a}=0\Rightarrow\int_V\left(\vec{\nabla}.\vec{A}\right)d\tau=0$ (From Divergence Theorem)

$\int_V\left(\vec{\nabla}.\vec{A}\right)d\tau=0\Rightarrow2k+l-3m=0$

Ans. 17: (a)

Solution: Since $\vec{A} = xz^2\hat{i} - yz^2\hat{j} + z\left(x^2 - y^2\right)\hat{k} \Rightarrow \vec{\nabla}.\vec{A} = z^2 - z^2 + \left(x^2 - y^2\right) = x^2 - y^2$

Thus

$$\int_V \left(\vec{\nabla}.\vec{A}\right)d\tau = \int_{x=-a}^{x=+a}\int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left(x^2 - y^2\right)dxdydz = \int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left[\frac{x^3}{3} - y^2x\right]_{-a}^{+a}dydz = \int_{y=-b}^{y=+b}\int_{z=0}^{z=c}\left[\frac{2}{3}a^3 - 2ay^2\right]dydz$$

$$\Rightarrow \int_V \left(\vec{\nabla}.\vec{A}\right)d\tau = \int_{z=0}^{z=c}\left[\frac{2}{3}a^3y - 2a\frac{y^3}{3}\right]_{-b}^{+b}dz = \int_{z=0}^{z=c}\left[\frac{4}{3}a^3b - \frac{4}{3}ab^3\right]dz = \frac{4}{3}abc\left[a^2 - b^2\right]$$

Ans. 18: (b)

Solution: $\vec{v} = \vec{\nabla}\times\vec{u} \Rightarrow \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = 0, \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = 0, \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = v_0$

Let $u_x = 0, u_z = 0 \Rightarrow \vec{u} = v_0x\hat{j}$

Ans. 19: (b)

Solution: Thus $\int_S \vec{v}.d\vec{a} = \int_S \left(\vec{\nabla}\times\vec{u}\right).d\vec{a} = \oint_{line} \vec{u}.d\vec{l}$

We have to take line integral around circle $x^2 + y^2 = r^2$ in $z = 0$ plane. Let use cylindrical coordinate and use $x = r\cos\phi,\ y = r\sin\phi \Rightarrow dy = r\cos\phi d\phi$.

$$\Rightarrow \oint_{line} \vec{u}.d\vec{l} = \oint_{line} v_0xdy = \int_0^{2\pi} v_0 \times r\cos\phi \times r\cos\phi d\phi$$

$$\Rightarrow \oint_{line} \vec{u}.d\vec{l} = v_0r^2\int_0^{2\pi}\cos^2\phi d\phi = v_0r^2\int_0^{2\pi}\left[\frac{1+\cos 2\phi}{2}\right]d\phi = \pi v_0r^2$$

$$\Rightarrow \int_S \vec{v}.d\vec{a} = \int_S \left(\vec{\nabla}\times\vec{u}\right).d\vec{a} = \oint_{line} \vec{u}.d\vec{l} = \pi v_0r^2$$

Ans. 20: (c)

Solution: $\vec{\nabla}\times\vec{F} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & 2xy & 0 \end{pmatrix} = 0$

Thus the force $\vec{F}(x,y)=k\left[\left(x^2+y^2\right)\hat{i}+2xy\ \hat{j}\right]$ is conservative. So work done is independent of paths.

Along line OD, $y=x\Rightarrow dy=dx$

Since $d\vec{l}=dx\hat{i}+dy\hat{j}\Rightarrow\vec{F}.d\vec{l}=\left[\left(x^2+y^2\right)dx+2xy\ dy\right]=\left[\left(x^2+x^2\right)dx+2x^2dx\right]=4x^2dx$

$$\int_{OD}\vec{F}.d\vec{l}=\int_{x=0}^{1}4x^2dx=\frac{4}{3}$$

Ans. 21: (a)

Solution: Since $\vec{\nabla}\times\vec{F}=\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-y^2 & 2xy & 0\end{pmatrix}=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(2y+2y)=4y\hat{k}$

Using the Stokes' theorem

$$\oint_{OPQRO}\vec{F}.d\vec{l}=\int_S\left(\vec{\nabla}\times\vec{F}\right).d\vec{a}=\int_0^a\int_0^b\left(4y\hat{k}\right).\left(-dxdy\hat{k}\right)=-4\int_0^a\int_0^b ydxdy=-4a\int_0^b ydy=-2ab^2$$

Ans. 22: (a)

Solution: For an arbitrary test function $f(x)$, consider the integral $\int_{-\infty}^{\infty}f(x)\delta(kx)dx$.

Changing variables, we let $y=kx$, so that $x=y/k\,x$, and $dx=\frac{1}{k}dy$. If k is positive, the integration still runs from $-\infty$ to $+\infty$, but if k is negative, then $x=\infty$ implies $y=-\infty$, and vice versa, so the order of limit is reversed. Restoring the "proper" order costs a minus sign. Thus

$$\int_{-\infty}^{\infty}f(x)\delta(kx)dx=\pm\int_{-\infty}^{\infty}f(y/k)\delta(y)\frac{dy}{k}=\pm\frac{1}{k}f(0)=\frac{1}{|k|}f(0)$$

(The lower signs apply when k is negative, and we account for this neatly by putting absolute value bars around the final k, as indicated.) Under the integral sign, then, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx \Rightarrow \delta(kx) = (1/|k|)\delta(x)$$

Ans. 23: (a)

Solution: Answer would be 0, because the spike would then be outside the domain of integration.

Ans. 24: (d)

Solution: $J = \int_v (r^2+2)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = \int_v (r^2+2)4\pi\delta^3(r)\,d\tau = 4\pi(0+2) = 8\pi$

Ans. 25: (c)

Solution: The delta function picks out the value of x^3 at the point $x = 2$ so the integral is $2^3 = 8$.

NAT (Numerical Answer Type)

Ans. 26: 0

Solution: $x^2 + y^2 = 1 \Rightarrow xdx = -ydy$ and $d\vec{l} = dx\hat{x} + dy\hat{y}$

$$\Rightarrow \vec{F}.d\vec{l} = \frac{xdx}{\sqrt{x^2+y^2}} + \frac{ydy}{\sqrt{x^2+y^2}} = 0 \quad (\because xdx = -ydy) \Rightarrow \int_A^B \vec{F}\cdot d\vec{l} = 0$$

Ans. 27: 3

Solution: $\oint_S \vec{r}.\hat{n}ds = \int_V (\vec{\nabla}.\vec{r})d\tau = 3\int_V d\tau = 3\pi a^2 h$

Ans. 28: 4

Solution: In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

$\because A_r = 2r\cos^2\phi,\ A_\phi = 3r^2\sin z\ ,\ A_z = 4z\sin^2\phi$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\times 2r\cos^2\phi\right) + \frac{1}{r}\frac{\partial\left(3r^2\sin z\right)}{\partial\phi} + \frac{\partial\left(4z\sin^2\phi\right)}{\partial z}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4\left(\cos^2\phi + \sin^2\phi\right) = 4$$

Ans. 29: 0.5

Solution: $\vec{u} = a\hat{i} + b\hat{n}$, $\vec{v} = a\hat{n} + b\hat{i}$

$$\Rightarrow \vec{u}\cdot\vec{v} = \left(a\hat{i} + b\hat{n}\right)\cdot\left(a\hat{n} + b\hat{i}\right) \Rightarrow \left|\vec{u}\right|\left|\vec{v}\right|\cos 60 = a^2\hat{i}\cdot\hat{n} + ab + ba + b^2\hat{n}.\hat{i}$$

$$\left(\sqrt{a^2+b^2+2ab\cos 120}\right)^2\cdot\cos 60 = a^2\cos 120 + 2ab + b^2\cos 120$$

$$\left(a^2+b^2-2ab\times\frac{1}{2}\right)\cdot\cos 60 = -\frac{1}{2}\left(a^2+b^2\right)+2ab = \frac{1}{2}\left(a^2+b^2\right)-\frac{ab}{2} = -\frac{1}{2}\left(a^2+b^2\right)+2ab$$

$$\Rightarrow a^2+b^2 = \frac{5ab}{2} \Rightarrow b = \frac{a}{2}$$

Ans. 30 : 3

Solution : Since $\oint_S \vec{A}.d\vec{a} = \int_V \left(\vec{\nabla}.\vec{A}\right)d\tau \Rightarrow \int_V \left(\vec{\nabla}.\vec{r}\right)d\tau = 3V$

MSQ (Multiple Select Questions)

Ans. 31: (a), (b) and (d)

Solution: Since $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Path (1) consists of two parts. Along the "horizontal" segment $dy = dz = 0$, so

(i) $d\vec{l} = dx\hat{x}$, $y = 1$, $\vec{A}\cdot d\vec{l} = y^2 dx = dx$, so $\int \vec{A}\cdot d\vec{l} = \int_1^2 dx = 1$

On the "vertical" stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\hat{y}$, $x = 2$, $\vec{A}\cdot d\vec{l} = 2x(y+1)dy = 4(y+1)dy$, so $\int \vec{A}\cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10$.

By path (1), then, $\int_a^b \vec{A}\cdot d\vec{l} = 1+10 = 11$

Meanwhile, on path (2) $x = y$, $dx = dy$, and $dz = 0$, so

$$d\vec{l} = dx\hat{x} + dx\hat{y},\quad \vec{A}\cdot d\vec{l} = x^2 dx + 2x(x+1)dx = \left(3x^2+2x\right)dx$$

so $$\int_a^b \vec{A}\cdot d\vec{l} = \int_1^2 \left(3x^2+2x\right)dx = \left(x^3+x^2\right)\Big|_1^2 = 10$$

For the loop that goes out (1) and back (2), then, $\oint \vec{A}\cdot d\vec{l} = 11-10 = 1$

1. Linear Algebra and Matrices

The main structures of linear algebra are vector spaces. A vector space over a field F is a set V, together with two binary operation. Elements of V are called *vectors* and elements of F are called *scalars*.

The first operation is vector addition; take any two vectors ϕ and ψ and output iss a third vector $\phi+\psi$. The second operation; take any scalar a and any vector ϕ and output is a new vector vector $a\phi$. In view of the first example, where the multiplication is done by rescaling the vector ϕ by a scalar a, the multiplication is called scalar multiplication of ϕ by a. The operations of addition and multiplication in a vector space satisfy the following axioms. In the list below, let ϕ, ψ and χ be arbitrary vectors in V, and a and b scalars in F.

The linear vector space consists of two sets of elements and two algebraic rules.

- A set of vectors $\psi, \phi, \chi, ...$ and set of scalars *a, b, c*
- A rule for vector addition and rule for scalar multiplication

(i) Addition rule

- If ψ and ϕ are vectors of elements of a space, their sum $\psi+\phi$ is also vector of the same space.

  Commutativity: $\psi+\phi=\phi+\psi$

  Associativity: $(\psi+\phi)+\chi=\psi+(\phi+\chi)$

  Existance of a symmetric and inverse vector: $\psi+(-\psi)=(-\psi)+\psi=0$

(ii) Multiplication rule

- The product of scalar with a vector gives another vector, If ψ and ϕ are two vectors of the space, any linear combination $a\psi+b\phi$ is also a vector of the space, where a and b being scalars.
- Distributivity with respect to addition:

  $a(\psi+\phi)=a\psi+a\phi; \quad (a+b)\psi=a\psi+b\psi$

- Associativity with respect to multiplication of scalars.

  $a(b\psi)=(ab)\psi$

- For each element ψ there must exist a unitary element I and a no. zero scalar such that $I\cdot\psi=\psi\cdot I=\psi$ $\quad O\cdot\psi=\psi\cdot O=O$

1.1 Linear Dependence and Dimensionality of a Vector Space

- A set of N vectors $\phi_1,\phi_2,...,\phi_N$ is said to be linearly independent if and only if the solution of the equation. $\sum_{i=1}^{N}a_i\phi_i=0$ if $a_1=a_2=a_3=a_4=0$ otherwise $\phi_1,\phi_2,....,\phi_N$ is said to be linear dependent.
- The dimension of a space vector is given by the maximum number of linearly independent vectors the space can have.
- The maximum number of linearly independent vectors a space has is $N(\phi_1,\phi_2,....,\phi_N)$, this space is said to be N dimensional. In this case any vector ψ of the vector space can be expressed as linear combination.

$$\psi=\sum_{i=1}^{N}a_i\phi_i$$

Inner product

- The inner product of two function $\psi_i(x,y,z)\psi_j(x,y,z)$ is 3 dimensional is defined or $(\psi_i,\psi_j)=\int(\psi_i^*\psi_j dxdydz)$
- Property of inner product

  (a) $(\psi,\phi)=(\phi,\psi)^*$

  $(\psi,a\phi_2+b\phi_2)=a(\psi,\phi_1)+b(\psi,\phi_2)$

  $(\psi,\psi)=|\psi|^2\geq 0$

  The norm (length of vector) is defined $(\psi,\psi)^{\frac{1}{2}}$

  If $(\psi_i,\psi_j)=\delta_{i,j}$ then vectors are said to form an orthonormal set

1.2 Properties of Matrices

a **matrix** (plural **matrices**) is a rectangular array of numbers, symbols, or expressions, arranged in *rows* and *columns*. The individual items in a matrix are called its *elements* or *entries*

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & a_{ij} & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In matrix A has m rows and n column a_{ij} is the element of i^{th} row and j^{th} column. The matrix A has m rows and n column, so the matrix is said to $m\times n\ (m\,by\,n)$.

If number of row and number of column in matrix is equal then matrix is said to be square matrix.

Operation of matrix

(a) Summation of matrices

In mathematics, **matrix addition** is the operation of adding two matrices by adding the corresponding entries together. The usual matrix addition is defined for two matrices of the same dimensions. The sum of two $m\times n$ (pronounced "m by n") matrices A and B, denoted by $A+B$, is again an $m\times n$ matrix computed by adding corresponding elements.

If $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$ then

$$A+B=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

(a) Direct sum

Another operation, which is used less often, is the direct sum (denoted by $\oplus$). Note the also denoted $\oplus$; the context should make the usage clear. The direct sum of any pair of matrices **A** of size $m \times n$ and **B** of size $p \times q$ is a matrix of size $(m+p)\times(n+q)$. The direct sum of matrices is a special type of block matrix $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

Multiplication of Matrix

(a) Scalar multiplication

The left multiplication of a matrix A with a scalar λ gives another matrix λA of the same size as A. The entries of λA are given by $\lambda(a_{ij}) = (\lambda a_{ij}) \Rightarrow \lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$

Right Scalar Multiplication

Similarly, the right multiplication of a matrix A with a scalar λ is defined to be $(a_{ij})\lambda = (\lambda a_{ij})$

(b) **Multiplication of two matrix**

Assume two matrices are to be multiplied (the generalization to any number is discussed below).If A is an $n \times m$ matrix, and B is an $m \times p$ matrix, the result AB of their multiplication is an $n \times p$ matrix defined only if the number of columns m in A is equal to the number rows m in B.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}$$

(where *necessarily* the number of columns in A equals the number of rows in B equals m) the matrix product **AB** is defined by

$$AB=\begin{bmatrix}(AB)_{11} & (AB)_{12} & \cdots & (AB)_{1p}\\ (AB)_{21} & (AB)_{22} & \cdots & (AB)_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ (AB)_{n1} & (AB)_{n2} & \cdots & (AB)_{np}\end{bmatrix} \text{ where } (AB)_{ij}=\sum_{k=1}^{m}a_{ik}b_{kj}$$

(c) Direct product (tensor product)

The direct product is defined for general matrices .if A is $n\times n$ square matrix and B is $m\times m$ square matrix .then direct product $A\otimes B$ is $nm\times nm$ matrix.

$C=A\otimes B$ where matrix element of C is given by $c_{ik,jl}=a_{ij}b_{kl}$

where $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ and $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ then $C=A\otimes B==\begin{pmatrix}a_{11}B & a_{12}B\\ a_{21}B & a_{22}B\end{pmatrix}$

$$C=A\otimes B=\begin{bmatrix}a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12}\\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22}\\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12}\\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22}\end{bmatrix}$$

Derivative of matrix

The derivative of a matrix with respect to a variable is equal to derivative of each element with respect to x separately.

Integral of matrix

The integral of matrix with respect to a variable x is equal to the integration of each element with respect to x separately.

Transpose of matrix (A^T)

The transpose of an arbitrary A is written as A^T is obtained by interchanging corresponding rows into column of A i.e. if element of A is a_{ij} then element of A^T is a_{ji} .

Complex –conjugate matrix (A^*)

The complex conjugate of an matrix A is formed by taking the complex conjugate of each element .hence matrix element of A^* is a_{ij}^* .

Hermitian conjugate ($A^\dagger$)

The hermitian conjugate of matrix A is obtain by taking the complex conjugate of the matrix and then the transpose of the complex conjugate matrix.

If matrix element of A is a_{ij} then matrix element of $A^\dagger$ **is** a_{ji}^*

Example: $A = \begin{bmatrix} 2+3i & 4-5i \\ 3 & 4i \end{bmatrix}$

$$A^* = \begin{bmatrix} 2-3i & 4+5i \\ 3 & -4i \end{bmatrix}, \quad A^\dagger = (A^*)^T = \begin{bmatrix} 2-3i & 3 \\ 4+5i & -4i \end{bmatrix}$$

Cofactor matrix (A^c)

The cofactor matrix A is matrix constructed by cofactor element a_{ij} which is denoted by A^{ij} which can be defined as $(-1)^{i+j} M_{ij}$ where M_{ij} are minor of element a_{ij} .

So $A^c = \begin{bmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{bmatrix}$

Adjoint of matrix

Transpose of cofactor matrix A^c is adjoint of matrix

$adj(A) = (A^c)^T$ if $adj(A) = A$ then it is said to be self adjoint matrix .

Inverse of matrix $\left(A^{-1}\right)$

If A is matrix and $A^{-1}A = I$ then A^{-1} is said to inverse of matrix A (where I) is identity matrix.

$A^{-1} = \dfrac{adjA}{|A|}$ where $|A|$ is determinant of matrix A. Inverse of A exist if $|A| \neq 0$.

Example: Find the inverse of $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution: $|A| = -1$, $adj(A) = (A^c)^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$$A^{-1} = \frac{adjA}{|A|} = \frac{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

one can check $A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.3 Eigen value problem

Let A be $n\times n$ square matrix. Consider the equation $AX_i = \lambda_i X_i$ where λ_i is scalar and X_i is vector. For every λ_i there corresponds $X_i \neq 0$ which satisfies the equation $AX_i = \lambda_i X_i$.

Then X_i is said to be Eigen vector corresponding to Eigen value λ_i of matrix A.

Characteristic Equation is given by determinant $A-\lambda I = 0$ i.e. $|A-\lambda I| = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$ which is polynomial of λ of order n which can be

- Represented by $D(\lambda)=0$ and Solution (roots) of characteristic equation is known as Eigen value.
- For an $n\times n$ matrix has at least one Eigen value and at most n numerically different Eigen values
- A repeated Eigen value is said to degenerate Eigen value. For degenerate Eigen value there is different Eigen vectors for same Eigen value
- Non repeated Eigen values are non degenerate Eigen values. For non degenerate Eigen value there is different Eigen vectors for different Eigen value .
- Sum of Eigen values are equal to trace of matrix $\sum_i \lambda_i = trace(A) = \sum_i^n a_{ii}$ trace of matrix is sum of diagonal element
- Product of Eigen values are equal to determinant of matrix $\prod_i^n \lambda_i = |A|$
- Eigen vectors correspond to different Eigen value will always independent.
- Eigen vectors correspond to same Eigen value may or may not be independent.

Example: Find the Eigen value and Eigen vector of matrix $A=\begin{bmatrix}2&0&0\\0&0&1\\0&1&0\end{bmatrix}$

For Eigen value $(A-\lambda I)=0\begin{vmatrix}2-\lambda&0&0\\0&-\lambda&1\\0&1&-\lambda\end{vmatrix}=0$

$$(2-\lambda)\{\lambda^2-1\}=0 \quad \Rightarrow \quad \lambda=2,\ \lambda^2-1=0\Rightarrow\ \lambda=\pm1$$

Eigen value is given by $\lambda_1=-1\ \lambda_2=1,\ \lambda_3=2,$ .where $\lambda_1=-1$ is non degenerate Eigen value all Eigen values are different so it is non degenerate .

Eigen vector can be determined by the equation $AX_i=\lambda_i X_i$.

$$\begin{bmatrix}2&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\lambda_i\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

- For $\lambda_1=-1 \quad \Rightarrow\begin{bmatrix}2&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=-1\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

which will give equation $2x_1=-x_1$ and $x_2=-x_3$ so $x_1=0$ and $x_2=-x_3$

$X_1=\begin{bmatrix}0\\x_2\\-x_2\end{bmatrix}$, Normalized Eigen value can be determined by relation $X_1^{\,T}X_1=1$ which will give

$x_2=\dfrac{1}{\sqrt{2}}$ so normalized Eigen vector corresponds to $\lambda_1=-1$ is $X_1=\dfrac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}$

- For $\lambda_2 = 1 \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ which will give equation $x_1 = 0$ and $x_2 = x_3$ so Eigen vector is given by $\begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix}$ for any value of x_2 one can find Eigen vector corresponds to $\lambda_2 = 1$ From orthogonal relation $X_2^T X_2 = 1$ so $X_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

- For $\lambda_2 = 2 \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ which will give $x_1 = x_1$ and $x_3 = 2x_2$ and $x_2 = 2x_3$ so $x_2 = x_2 = 0$ Eigen vector will be $X_3 = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$ value of x_1 can be determine from orthogonal condition $X_3^T X_3 = 1$ $x_1 = 1$ so $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Example: Find The Eigen value and normalized Eigen vector of matrix $A=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

For Eigen value $(A-\lambda I)=0 \quad \Rightarrow \begin{vmatrix}1-\lambda&0&0\\0&-\lambda&1\\0&1&-\lambda\end{vmatrix}=0$

$$(1-\lambda)\{\lambda^2-1\}=0 \quad \Rightarrow \quad \lambda=1,\ \lambda^2-1=0 \Rightarrow \lambda=\pm 1$$

Eigen value is given by $\lambda_1=-1\ \lambda_2=1,\ \lambda_3=1$ where $\lambda_1=-1$ is non degenerate Eigen value $\lambda_2=\lambda_3=1$ is doubly degenerate Eigen value.

Eigen vector can be determined by the equation $AX_i=\lambda_i X_i$.

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\lambda_i\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

- For $\lambda_1=-1 \quad \Rightarrow \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=-1\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Which will give equation $x_1=-x_1$ and $x_2=-x_3$ So $x_1=0$ and $x_2=-x_3$

$X_1=\begin{bmatrix}0\\x_2\\-x_2\end{bmatrix}$ normalized Eigen value can be determined by relation $X_1^T X_1=1$

which will give $x_2=\frac{1}{\sqrt{2}}$ so normalized

Eigen vector corresponds to $\lambda_1=-1$ is $X_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}$

- For $\lambda_2 = \lambda_3 = 1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ which will give equation $x_1 = x_1$ and $x_2 = x_3$

So, Eigen vector is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}$ for any value of x_1 and x_2 one can find Eigen vector corresponds to $\lambda_2 = \lambda_3 = 1$ but we need to find orthogonal Eigen vectors.

Let $x_1 = x_1$ and $x_2 = 0$ so $X_2 = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$ $X_2^T X_2 = 1$ will give $x_1 = 1$ so $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now we must find third Eigen vector which will satisfied equation $x_1 = x_1$ and $x_2 = x_3$ and orthogonal to X_2 .so $X_2^T X_3 = 0$. Which will give relation $x_1 = 0$ and $x_2 = x_3$

So $X_3 = \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix}$ and value of x_2 can be find with orthogonal relation $X_3^T X_3 = 1$ which will give

$x_2 = \frac{1}{\sqrt{2}}$ so $X_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

1.4 Different Types of Matrices and their properties

Unit matrix

A unit matrix is a diagonal matrix whose all diagonal elements is1.

$$\delta_{i,j} = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

Example: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, Eigen value of unit matrix is 1 and it is degenerate. The independent and orthonormal Eigen vector are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Symmetric matrix

A square matrix A is defined such that $A^T = A$. The matrix element of Symmetric matrix is given by $a_{ij} = a_{ji}$ .The Eigen value of symmetric matrix is always real.

Example: $A = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$.

The Eigen value of A matrix is given by $\lambda_1 = \frac{5+\sqrt{65}}{2}$ and $\lambda_2 = \frac{5-\sqrt{65}}{2}$ which is real.

Anti-symmetric (skew) matrix

If a square matrix A is defined such that $A^T = -A$. The matrix element of antisymetric is given by $a_{ij} = -a_{ji}$. The Eigen value of skew symmetric matrix is purely imaginary or zero. The diagonal element of skew symmetric matrix is always zero.

Example: $A = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}$

The Eigen value of $\lambda_1 = 4i$ and $\lambda_1 = -4i$

Orthogonal matrix

If a square matrix A is defined such that $A^T = A^{-1}$. The matrix element of an orthogonal matrix has value $+1$ and -1 . The eigen value of an orthogonal matrix A are real or complex in pair and have absolute value1.

Example: $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

The determinant of matrix A i.e. $|A| = 1$ and the eigen value of matrix is $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$ which have absolute value 1 i.e. $|\lambda_1| = 1$ and $|\lambda_2| = 1$.

Hermitian matrix

A square matrix A is said to be Hermitian matrix if $A^{\dagger} = A$ where $A^{\dagger} = (A^{*})^{T}$ that means matrix element is given by $a_{ij} = a_{ji}^{*}$ .Eigen value of Hermitian matrix is real.

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$

The Eigen value of hermitian matrix A is $-1,1,2$ which is real.

Skew Hermitian matrix

A square matrix A is said to be Hermitian matrix if $A^{\dagger} = -A$ where $A^{\dagger} = (A^{*})^{T}$ that means matrix element is given by $a_{ij} = -a_{ji}^{*}$. Eigen value of Hermitian matrix is purely imaginary or zero diagonal matrix of skew hermitian matrix is zero.

Example: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

The Eigen value of matrix A is $i, 0, -i$.

Unitary matrix

If a square matrix A is defined such that $A^{\dagger} = A^{-1}$. The matrix element an unitary matrix has absolute value 1 ie $|A| = 1$. The Eigen value of an unitary matrix A are real or complex in pair and have absolute value1.

Example: $A = \begin{pmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{pmatrix}$

The absolute determinant of matrix A is 1 ie. $|A| = 1$ the Eigen value of given matrix is given by $\lambda_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}$ and $\lambda_1 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$ which have absolute value 1 ie $|\lambda_1| = 1$ and $|\lambda_2| = 1$.

1.5 Cayley–Hamilton Theorem

Every square matrix satisfied its own characteristic equation. The characteristic equation of $n\times n$ square matrix A given by $|A-\lambda I|=0$ which is polynomial of order n (let say $\phi(\lambda)$)

$\phi(\lambda)=\phi_0 I+\phi_1\lambda+.....+\phi_{n-1}\lambda^{n-1}+\phi_n\lambda^n=0$ is characteristic equation .

From Cayley-Hamilton theorem $\phi(A)=\phi_0 I+\phi_1 A+.....+\phi_{n-1}A^{n-1}+\phi_n A^n=0$

Example: illustrate the Cayley – Hamilton theorem for the matrix $A=\begin{pmatrix}1&2&0\\2&-1&0\\0&0&1\end{pmatrix}$

Solution: the characteristic equation is $\begin{vmatrix}1-\lambda&2&0\\2&-1-\lambda&0\\0&0&1-\lambda\end{vmatrix}=-5+5\lambda+\lambda^2-\lambda^3$

$\phi(\lambda)=-5+5\lambda+\lambda^2-\lambda^3=0$ the matrix A will satisfied the same equation

$$\phi(A)=-5+5A+A^2-A^3$$

$$\Rightarrow\begin{pmatrix}-5&0&0\\0&-5&0\\0&0&-5\end{pmatrix}+\begin{pmatrix}5&10&0\\10&-5&0\\0&0&5\end{pmatrix}+\begin{pmatrix}5&0&0\\0&5&0\\0&0&1\end{pmatrix}+\begin{pmatrix}-5&-10&0\\-10&5&0\\0&0&-1\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Example: find the inverse of matrix $A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ from Cayley -Hamilton theorem.

Solution: The characteristic equation is given by $\begin{vmatrix}0-\lambda&0&1\\0&1-\lambda&0\\1&0&0-\lambda\end{vmatrix}=0$

$\lambda^3-\lambda^2-\lambda+1=0$ from Cayley–Hamilton theorem A will satisfied characteristic equation .

$A^3-A^2-A+I=0$.

Multiply equation by A^{-1} one can get $A^2-A^1-I+A^{-1}=0\Rightarrow A^{-1}=A^2-A^1-I=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$

So for matrix $A = A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

1.6 Diagonalisation of Matrix

A square matrix A is called **diagonalizable** if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix. If V is a finite-dimensional vector space(one can find independent Eigen vector corresponds to Eigen value) , then a linear map $T : V \to V$; T is called **diagonalizable** if there exists a basis of V with respect to which T is represented by a diagonal matrix. **Diagonalization** is the process of finding a corresponding diagonal matrix for a diagonalizable matrix or linear map. A square matrix which is not diagonalizable is called *defective.*

Example: Consider a matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}$. This matrix has eigenvalues $\lambda_1 = 3, \lambda_2 = 2, \lambda_1 = 1$

A is a 3×3 matrix with 3 different eigenvalues; therefore one can find basis in which , it is diagonalizable.(Note that if there are exactly n distinct eigenvalues in an $n \times n$ matrix then this matrix is diagonalizable.)

These eigenvalues are the values that will appear in the diagonalized form of matrix A, so by finding the eigenvalues of A we have diagonalized it.

The eigenvectors of A corresponds to eigen value respectively $\lambda_1 = 3, \lambda_2 = 2, \lambda_1 = 1$ are

$$X_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Now, let P be the matrix with these eigenvectors as its columns: $P = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

Note: There is no preferred order of the eigenvectors in P; changing the order of the eigenvectors in P just changes the order of the eigenvalues in the diagonalzed form of A.

Then P diagonalizes A, as a simple computation confirms, having calculated P^{-1} using any suitable method:

$$P^{-1}AP = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the eigenvalues $\lambda_1 = 3, \lambda_2 = 2, \lambda_1 = 1$ appear in the diagonal matrix.

Diagonalization can be used to compute the powers of a matrix A efficiently, provided the matrix is diagonalizable. Suppose we have found that

$$P^{-1}AP = D$$

is a diagonal matrix. Then, as the matrix product is associative,

$$A^K = (PDP^{-1})^K = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1}).....(PDP^{-1})$$

$PD^K P^{-1}$ and the latter is easy to calculate since it only involves the powers of a diagonal matrix.

1.7 Function of Matrix

If the matrix A is diagonalizable, then we can find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Applying the power series definition to this decomposition, we find that $f(A)$ is defined by

$$f(A) = P\begin{bmatrix} f(d_1) & 0 & \cdots & 0 \\ 0 & f(d_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(d_n) \end{bmatrix}P^{-1}$$

where $d_1, d_2....d_n$ denote the diagonal entries of D.

Example: If matrix $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then find e^A.

Solution: For matrix A Eigen value is λ_1, λ_2 and Eigen vectors are respective $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

$$P=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \quad \text{and} \quad P^{-1}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$e^{A}=Pf(A)P^{-1}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}e^{\lambda_1} & 0\\0 & e^{\lambda_2}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}e^{\lambda_1} & 0\\0 & e^{\lambda_2}\end{bmatrix}$$

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Multiple Choice Questions

Q1. The eigen value of matrix $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$ is

(a) $\lambda=1,\ 0,\ 2$ (b) $\lambda=-1,\ 2,\ 2$ (c) $\lambda=0,\ 0,\ 3$ (d) $\lambda=1,\ 1,\ 1$

Q2. The eigen value of matrix $A=\begin{pmatrix}1 & \sqrt{8} & 0\\ \sqrt{8} & 1 & \sqrt{8}\\ 0 & \sqrt{8} & 1\end{pmatrix}$ is

(a) $\lambda=1,\ 0,\ 2$ (b) $\lambda=-1,\ 2,\ 2$ (c) $\lambda=-3,\ 1,\ 5$ (d) $\lambda=1,\ 1,\ 1$

Q3. The eigen value of matrix $A=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ is

(a) $\lambda=-1,\ 0,\ 1$ (b) $\lambda=0,\ -2,\ 2$ (c) $\lambda=0,\ 0,\ 0$ (d) $\lambda=-\sqrt{2},\ 0,\ \sqrt{2}$

Q4. The eigen value of matrix $A=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$ is

(a) $\lambda=-1,\ -1,\ 2$ (b) $\lambda=0,\ -2,\ 2$ (c) $\lambda=0,\ 0,\ 0$ (d) $\lambda=-\sqrt{2},\ 0,\ \sqrt{2}$

Q5. The degenerate eigen value of matrix $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Q6. The eigen vector of matrix $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$ corresponding to eigen value 0 is

(a) $\begin{bmatrix}1\\0\\0\end{bmatrix}$ (b) $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ (c) $\begin{bmatrix}1\\0\\1\end{bmatrix}$ (d) $\begin{bmatrix}1\\-1\\0\end{bmatrix}$

Q7. The eigen vector of matrix $A=\begin{pmatrix}1 & \sqrt{8} & 0\\ \sqrt{8} & 1 & \sqrt{8}\\ 0 & \sqrt{8} & 1\end{pmatrix}$ corresponding to eigen value 5 is

(a) $\begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}$ (b) $\begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix}$ (c) $\begin{bmatrix}\sqrt{2}\\ 1\\ 1\end{bmatrix}$ (d) $\begin{bmatrix}-\sqrt{2}\\ 1\\ 1\end{bmatrix}$

Q8. The eigen vectors of matrix $A=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ corresponding to eigen values $-\sqrt{2},\ 0,\ \sqrt{2}$ are

respectively

(a) $\begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}, \begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}$ (b) $\begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}, \begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix}$

(c) $\begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix}, \begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$ (d) $\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}, \begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix}, \begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}$

Q9. If one of the eigen vector of matrix $A=\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$ corresponding to eigen value -1 is $\begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$ then

other orthogonal eigen vector for same eigen value is

(a) $\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$ (b) $\begin{bmatrix}-4\\ 2\\ 2\end{bmatrix}$ (c) $\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$ (d) $\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$

Q10. If one of the eigen vector of matrix $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ corresponding to eigen value 0 is $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ then other orthogonal eigen vector for same eigen value is

(a) $\begin{bmatrix}-2\\1\\1\end{bmatrix}$ (b) $\begin{bmatrix}0\\2\\-2\end{bmatrix}$ (c) $\begin{bmatrix}1\\0\\1\end{bmatrix}$ (d) $\begin{bmatrix}1\\-1\\0\end{bmatrix}$

Q11. A square matrix 3×3 is given by $A=\begin{bmatrix}2&0&0\\0&1&1\\0&1&1\end{bmatrix}$ is diagonalized in eigenvector of matrix $S=\begin{bmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$. Which one of the following is matrix A in the diagonal form in the basis of S?

(a) $\begin{bmatrix}2&0&0\\0&2&0\\0&0&0\end{bmatrix}$ (b) $\begin{bmatrix}0&0&0\\0&2&0\\0&0&2\end{bmatrix}$ (c) $\begin{bmatrix}2&0&0\\0&0&0\\0&0&2\end{bmatrix}$ (d) $\begin{bmatrix}1&0&2\\0&1&0\\0&0&2\end{bmatrix}$

Q12. Consider the matrix $M=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$. The eigenvalues of M are

(a) 0, 1, 2 (b) 0, 0, 3 (c) 1, 1, 1 (d) – 1, 1, 3

Q13. Consider the matrix $M=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$. The exponential of M simplifies to (I is the 3 × 3 identity matrix)

(a) $e^M=I+\left(\frac{e^3-1}{3}\right)M$ (b) $e^M=I+M+\frac{M^2}{2!}$

(c) $e^M=I+3^3M$ (d) $e^M=(e-1)M$

Q14. The eigenvalues of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are

(a) 5, 2, -2 (b) -5, -1, -1 (c) 5, 1, -1 (d) -5, 1, 1

Q15. The eigen values of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ are

(a) $(1,4,9)$ (b) $(0,7,7)$ (c) $(0,1,13)$ (d) $(0,0,14)$

Q16. The eigenvalues of the antisymmetric matrix, $A = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$ where n_1, n_2 and n_3 are the components of a unit vector, are

(a) $0, i, -i$ (b) $0, 1, -1$ (c) $0, 1+i, -1, -i$ (d) $0, 0, 0$

Q17. Consider the matrix $M = \begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$. The eigenvalues of M are

(a) $-5, -2, 7$ (b) $-7, 0, 7$ (c) $-4i, 2i, 2i$ (d) $2, 3, 6$

Q18. The column vector $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ is a simultaneous eigenvector of

$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, if

(a) $b = 0$ or $a = 0$ (b) $b = a$ or $b = -2a$

(c) $b = 2a$ or $b = -a$ (d) $b = a/2$ or $b = -a/2$

Q19. Let α and β be complex numbers. Which of the following sets of matrices forms a group under matrix multiplication?

(a) $\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$, where $\alpha\beta \neq 1$

(c) $\begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}$, where $\alpha\beta^*$ is real (d) $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$, where $|\alpha|^2 + |\beta|^2 = 1$

Q20. A 3×3 matrix M has $Tr[M] = 6$, $Tr[M^2] = 26$ and $Tr[M^3] = 90$. Which of the following can be a possible set of eigenvalues of M ?

(a) $\{1,1,4\}$ (b) $\{-1,0,7\}$ (c) $\{-1,3,4\}$ (d) $\{2,2,2\}$

Q21. The matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is

(a) Orthogonal (b) symmetric (c) anti-symmetric (d) Unitary

Q22. If H is Hermitian matrix then matrix $A = \exp(iH)$ is

(a) Hermitian (b) Unitary (c) Skew Hermitian (d) Identity

Q23. The matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is diagonalize in the basis of unitary matrices U and get the diagonalise matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ then matrix U is

(a) $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Q24. Given a 2×2 unitary matrix U satisfying $U^{\dagger}U = UU^{\dagger} = 1$ with $\det U = e^{i\varphi}$, one can construct a unitary matrix $V\left(V^{\dagger}V = VV^{\dagger} = 1\right)$ with $\det V = 1$ from it by

(a) multiplying U by $e^{-i\varphi/2}$

(b) multiplying any single element of U by $e^{-i\varphi}$

(c) multiplying any row or column of U by $e^{-i\varphi/2}$

(d) multiplying U by $e^{-i\varphi}$

Q25. Consider an $n\times n(n>1)$ matrix A, in which A_{ij} is the product of the indices i and j (namely $A_{ij} = ij$). The matrix A

(a) has one degenerate eigevalue with degeneracy $(n-1)$

(b) has two degenerate eigenvalues with degeneracies 2 and $(n-2)$

(c) has one degenerate eigenvalue with degeneracy n

(d) does not have any degenerate eigenvalue

Q26. Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P. Which of the following statements is NOT TRUE?

(a) Det$A$ = Det$B$ (b) Trace of A = Trace of B

(c) A and B have the same eigenvectors (d) A and B have the same eigenvalues

Q27. A 3×3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalues of the matrix is

(a) 18 (b) 12 (c) 9 (d) 6

Q28. The inverse of matrix $M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is

(a) $M - 1$ (b) $M^2 - 1$ (c) $I - M^2$ (d) $I - M$

MSQ (Multiple Select Questions)

Q29. For matrix $A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, which of the following statements are true?

(a) The degenerate eigen value is 1

(b) One of the eigen vector corresponding to degenerate eigen value is $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(c) The eigen vector corresponding to nondegenerate eigen value is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(d) The eigen vector corresponding to nondegenerate eigen value is $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Q30. For matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which of the following statements are true?

(a) The degenerate eigen value is 0

(b) One of the eigen vector corresponding to degenerate eigen value is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(c) One of the eigen vector corresponding to degenerate eigen value is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) The eigen vector corresponding to nondegenerate eigen value is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Q31. For matrix $A=\begin{pmatrix}5 & 0 & \sqrt{3}\\ 0 & 3 & 0\\ \sqrt{3} & 0 & 3\end{pmatrix}$, which of the following statements are true?

(a) The eigen value are $2,\ 3,\ 6$

(b) The eigen value are $2,\ 4,\ 5$

(c) Eigen vector corresponding eigen value 2 is $\begin{bmatrix}1\\ 0\\ -\sqrt{3}\end{bmatrix}$

(d) Eigen vector corresponding eigen value 2 is $\begin{bmatrix}\sqrt{3}\\ 0\\ 1\end{bmatrix}$

Q32. Which one of following is correct

(a) If $A^{\dagger}=A$ and $B^{\dagger}=-B$ Then $AB+BA$ is skew Hermitian

(b) If $A^{\dagger}=A$ and $B^{\dagger}=-B$ Then $AB+BA$ is Hermitian

(c) If $A^{\dagger}=A$ and $B^{\dagger}=-B$ $i(AB+BA)$ is skew Hermitian

(d) $A^{\dagger}=A$ and $B^{\dagger}=-B$ $i(AB+BA)$ is Hermitian

Q33. Which of the following is correct for matrix $A=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$

(a) It is its own inverse

(b) It is its own transpose

(c) It has eigen value ± 1

(d) It is orthogonal matrix.

NAT (Numerical Answer Type)

Q34. The degenerate eigenvalue of the matrix $\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ is (your answer should be an integer)

_____

Q35. The minimum eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is……

Q36. The degenerate eigen value of matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is given by ……

Solutions

Multiple Choice Questions

Ans. 1: (a)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 0 & 1-\lambda\\ 0 & 1-\lambda & 0\\ 1 & 0 & 1-\lambda\end{vmatrix}=(1-\lambda)\left[(1-\lambda)^2-0\right]+0+1\left[-(1-\lambda)\right]=0$$

$$\Rightarrow -\lambda^3+3\lambda^2-2\lambda=0 \Rightarrow \lambda=0,1,2$$

Ans. 2: (c) $\lambda=-3,\ 1,\ 5$

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & \sqrt{8} & 0\\ \sqrt{8} & (1-\lambda) & \sqrt{8}\\ 0 & \sqrt{8} & (1-\lambda)\end{vmatrix}=(1-\lambda)\left[(1-\lambda)^2-8\right]+\sqrt{8}\left[\sqrt{8}(1-\lambda)\right]$$

$$-\lambda^3+3\lambda^2-3\lambda+1=0 \Rightarrow \lambda=-3,1,5$$

Ans. 3: (d) $\lambda=-\sqrt{2},\ 0,\ \sqrt{2}$

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{vmatrix}=0\Rightarrow -\lambda\left(\lambda^2-1\right)+1\left[(\lambda)\right]=0\Rightarrow -\lambda^3+2\lambda=0$$

eigenvalue $\lambda=0,\sqrt{2},-\sqrt{2}$

Ans. 4: (a) $\lambda=-1,\ -1,\ 2$

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0\Rightarrow -\lambda\left[\left(\lambda^2-1\right)\right]+1(1+\lambda)+1(1+\lambda)=0$$

$$\Rightarrow -\lambda^3+3\lambda+2=0 \Rightarrow \lambda=-1,-1,2$$

Ans. 5: (a)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)\left[\left(1-\lambda^2\right)-1\right]+1\left[1-(1-\lambda)\right]+1\left[1-(1-\lambda)\right]=0$$

$$\Rightarrow-\lambda^3+3\lambda^2=0\ \Rightarrow\lambda=0,0,3$$

Ans. 6: (b)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 0 & 1-\lambda\\ 0 & 1-\lambda & 0\\ 1 & 0 & 1-\lambda\end{vmatrix}=(1-\lambda)\left[(1-\lambda)^2-0\right]+0+1\left[-(1-\lambda)\right]=0$$

$$\Rightarrow-\lambda^3+3\lambda^2-2\lambda=0\ \Rightarrow\lambda=0,1,2$$

Eigen vector can be determine by $AX_1=\lambda X_1\Rightarrow\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\lambda\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

For $\lambda=0$,

$$\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\Rightarrow x_1+x_3=0,\ x_2=x_2,\ x_1+x_3=0\Rightarrow x_1=-x_3\ \text{and}\ x_2=0$$

$$X_1=\begin{bmatrix}x_1\\0\\-x_1\end{bmatrix}=x_1\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

From orthogonal relation $X_1^{\,T}X_1=1\Rightarrow x_1=\frac{1}{\sqrt{2}}\Rightarrow X_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

Ans. 7: (a)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & \sqrt{8} & 0 \\ \sqrt{8} & (1-\lambda) & \sqrt{8} \\ 0 & \sqrt{8} & (1-\lambda) \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2-8\right]+\sqrt{8}\left[\sqrt{8}(1-\lambda)\right]$$

$-\lambda^3+3\lambda^2-3\lambda+1=0 \Rightarrow \lambda=-3,1,5$

eigen vector can be determine by $AX_1=\lambda_1 X_1$

For $\lambda=5$,

$$\begin{bmatrix} 1 & \sqrt{8} & 0 \\ \sqrt{8} & 1 & \sqrt{8} \\ 0 & \sqrt{8} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=5\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1+\sqrt{8}x_2=5x_1, \sqrt{8}\,x_1+x_2+\sqrt{8}x_3=5x_2, \sqrt{8}x_2+x_3=5x_3$

$$\Rightarrow x_1=x_3, x_2=\sqrt{2}x_1 \quad X_1=\begin{bmatrix} x_1 \\ \sqrt{2}x_1 \\ x_1 \end{bmatrix} \Rightarrow X_1=x_1\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

Form orthogonality condition $x_1=\frac{1}{2}$, $X_1=\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

Ans. 8: (a)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}=0 \Rightarrow -\lambda\left(\lambda^2-1\right)+1\left[(\lambda)\right]=0 \Rightarrow -\lambda^3+2\lambda=0$$

eigenvalue $\lambda=0,\sqrt{2},-\sqrt{2}$

eigen vector can be determine by $AX_1=\lambda_1 X_1$

For $\lambda = -\sqrt{2}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\sqrt{2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_2 = -\sqrt{2}x_1,\ \ x_1 + x_3 = -\sqrt{2}x_2,\ \ x_2 = -\sqrt{2}x_3$$

$$\Rightarrow x_1 = x_3,\ x_2 = -\sqrt{2}x_1$$

$X_1 = \begin{bmatrix} x_1 \\ -\sqrt{2}x_1 \\ x_1 \end{bmatrix}$. From orthogonality condition, $X_1^T X_1 = 1 \quad \Rightarrow x_1 = \frac{1}{2} \Rightarrow X_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

For $\lambda = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_2 = 0,\ x_1 + x_3 = 0, \Rightarrow x_1 = -x_3 \Rightarrow X_2 = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix}$$

From orthogonality condition $X_2^T X_2 = 1 \Rightarrow x_1 = \frac{1}{\sqrt{2}} \Rightarrow X_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda = \sqrt{2}$,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \sqrt{2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow\ x_2 = \sqrt{2}x_1,\ x_1 + x_3 = \sqrt{2}x_2,\ x_2 = \sqrt{2}x_3 \Rightarrow x_1 = x_3$$

$\Rightarrow X_3 = \begin{bmatrix} x_1 \\ \sqrt{2}x_1 \\ x_1 \end{bmatrix}$. From orthogonality condition $X_3^T X_3 = 1 \Rightarrow x_1 = \frac{1}{2} \Rightarrow X_3 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

Ans. 9: (a)

Solution: For eigen value $|A - \lambda I| = 0$ $\qquad X_1 = \frac{1}{2}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left[\left(\lambda^2 - 1\right)\right] + 1(1+\lambda) + 1(1+\lambda) = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda + 2 = 0 \Rightarrow \lambda = -1, -1, 2$$

Eigen vector can be determine by $AX_1 = \lambda_1 X_1$

For $\lambda = -1$,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_2 + x_3 = -x_1,\ \ x_1 + x_3 = -x_2,\ x_1 + x_2 = -x_3 \Rightarrow x_1 + x_2 + x_3 = 0,$$

if $X_1 = \begin{bmatrix} -(x_2 + x_3) \\ x_2 \\ x_3 \end{bmatrix}$ if $\Rightarrow x_2 = k_1\ x_3 = k_2$

$$X_1 = \begin{bmatrix} -(k_1 + k_2) \\ k_1 \\ k_2 \end{bmatrix} \text{ if } k_1 = k_2$$

$$X_1 = k_1\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ from orthogonality condition } X_1^T X_1 = 1 \Rightarrow k_1 = \frac{1}{2}$$

$$X_1 = \frac{1}{2}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = -1$ similarly if $k_1 = 1, k_2 = -1$

Then $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $X_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Ans. 10: (a)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)\left[\left(1-\lambda^2\right)-1\right]+1\left[1-(1-\lambda)\right]+1\left[1-(1-\lambda)\right]=0$$

$\Rightarrow -\lambda^3+3\lambda^2=0 \Rightarrow \lambda=0,0,3$

eigen vector can be determine by $AX_1=\lambda_1 X_1$

For $\lambda=0$,

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\Rightarrow x_1+x_2+x_3=0,\ x_1+x_2+x_3=0,\ x_1+x_2+x_3=0$$

Now $x_1=-x_2-x_3$, if $x_2=-1, x_3=1 \Rightarrow x_1=0$

$$X_1=\begin{bmatrix}-x_2-x_3\\ x_2\\ x_3\end{bmatrix}\Rightarrow X_1=\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}=X_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$$

For $\lambda=0$, $X_2=\begin{bmatrix}-x_2-x_3\\ x_2\\ x_3\end{bmatrix}$ is orthogonal to X_1.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & -1\end{pmatrix}\begin{bmatrix}-x_2-x_3\\ x_2\\ x_3\end{bmatrix}=0\Rightarrow x_2=x_3\Rightarrow X_2=\begin{bmatrix}-2x_2\\ x_2\\ x_2\end{bmatrix}$$

From orthogonality condition, $X_2^T X_2=1\Rightarrow x_2=\frac{1}{\sqrt{6}}\Rightarrow X_2=\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$

Ans. 11: (c)

Solution: A diagonalised $= S^{-1}A\,S$ ….(i)

$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, and finding S^{-1} and putting in (i), we get

A diagonalized $= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Ans12: (b)

Solution: For eigen values $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = 0$

$(1-\lambda)\left((1-\lambda)^2-1\right)-(1-\lambda-1)+1(1-(1-\lambda))=0$

$(1-\lambda)\left(1+\lambda^2-2\lambda-1\right)+\lambda+\lambda=0 \Rightarrow \lambda^2-2\lambda-\lambda^3+2\lambda^2+2\lambda=0$

$\lambda^3-3\lambda^2=0 \Rightarrow \lambda^2(\lambda-3)=0 \Rightarrow \lambda=0,0,3$

For any $n\times n$ matrix having all elements unity eigenvalues are $0,0,0,...,n$.

Ans. 13: (a)

Solution: For e^M let us try to diagonalize matrix M using similarity transformation.

For $\lambda=3$, $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1+x_2+x_3=0$, $x_1-2x_2+x_3=0$, $x_1+x_2-2x_3=0$

$\Rightarrow -3x_2+3x_3=0$ or $x_2=x_3 \Rightarrow x_1=x_2=x_3=k$.

Eigen vector is $1\sqrt{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$ where $k=1$.

For $\lambda=0$,

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow x_1+x_2+x_3=0$$

Let $x_1=k_1, x_2=k_2$ and $x_3=k_1+k_2$. Eigen vector is $\begin{bmatrix}k_1\\k_2\\(k_1+k_2)\end{bmatrix}=1/\sqrt{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}$ where $k_1=k_2=1$.

Let $x_1=k_1, x_2=k_2$ and $x_3=-(k_1+k_2)$. Other Eigen vector $1/\sqrt{2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ where $k_1=1, k_2=-1$.

$$S=\begin{bmatrix}0&1&1\\-1&0&1\\1&-1&1\end{bmatrix}\Rightarrow S^{-1}=\begin{bmatrix}1&-2&1\\2&-1&-1\\1&-1&1\end{bmatrix}\Rightarrow D=S^{-1}MS, M=SDS^{-1}.$$

$$e^M=Se^DS^{-1}\Rightarrow e^D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&e^3\end{bmatrix}\Rightarrow e^M=1+\frac{(e^3-1)M}{3}$$

Ans. 14: (c)

Solution: The characteristic equation of the matrix A, $|A-\lambda I|=0$

$$\Rightarrow|A-\lambda I|=\begin{vmatrix}2-\lambda&3&0\\3&2-\lambda&0\\0&0&1-\lambda\end{vmatrix}=0\Rightarrow\lambda^3-5\lambda^2-\lambda+5=0\Rightarrow\lambda=5,1,-1$$

Ans. 15: (d)

Solution: For eigenvalues $|A-\lambda I|=0\Rightarrow\begin{bmatrix}1-\lambda&2&3\\2&4-\lambda&6\\3&6&9-\lambda\end{bmatrix}=0$

$$(1-\lambda)[(4-\lambda)(9-\lambda)-36]-2[2(9-\lambda)-18]+3[12-3(4-\lambda)]=0$$

$(1-\lambda)(4-\lambda)(9-\lambda)-36(1-\lambda)-4(9-\lambda)+36+9\lambda=0$

$\lambda^3-14\lambda^2=0 \Rightarrow \lambda^2(\lambda-14)=0 \Rightarrow \lambda=0,\ 0,\ 14$.

Ans. 16: (a)

Solution: $A=\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \Rightarrow -A^T=\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$

$\Rightarrow \lambda_1=0 \qquad \Rightarrow \lambda_2=-\sqrt{-n_1^2-n_2^2-n_3^2} \qquad \Rightarrow \lambda_3=\sqrt{-n_1^2-n_2^2-n_3^2}$

but $\sqrt{n_1^2+n_2^2+n_3^2}=1$ so, $\lambda_1=0,\ \lambda_2=L,\ \lambda_3=-L$

$A=-A^T$ (Antisymmetric). Eigenvalues are either zero or purely imaginary.

Ans. 17: (b)

Solution: $M=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix},\ M^+=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$

$M^+=M$

Matrix is Hermitian so roots are real and trace = 0.

$\lambda_1+\lambda_2+\lambda_3=0,\ \lambda_1\cdot\lambda_2\cdot\lambda_3=0 \Rightarrow \lambda=-7,0,7$

Ans. 18: (b)

Solution: Let $b=a$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix}=\begin{pmatrix} a \\ a \\ a \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix}=\begin{pmatrix} a \\ a \\ a \end{pmatrix}=\begin{pmatrix} a \\ a \\ a \end{pmatrix}$

Let $b=-2a$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}=\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}=\begin{pmatrix} -a \\ 2a \\ -a \end{pmatrix}=-1\begin{pmatrix} a \\ -2a \\ a \end{pmatrix}$

For other combination above relation is not possible.

Ans. 19: (d)

Solution: $\because \begin{vmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{vmatrix} = |\alpha|^2 + |\beta|^2 = 1$

Ans. 20: (c)

Solution: $Tr[M^2] = (-1)^2 + (3)^2 + (4)^2$ also $Tr[M^3] = (-1)^3 + (3)^3 + (4)^3 = 90$.

Ans. 21: (d)

Solution: Unitary $A^\dagger A = I$

Ans. 22: (b)

Solution: $A^\dagger = \exp(-iH) = (\exp iH)^{-1} = A^{-1}$

Ans. 23: (c)

Solution: Eigen value of matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is $\lambda = 2, 1, -1$.

corresponding eigen vectors are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

diagonalise matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ so U is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Ans. 24: (a)

Ans. 25: (a)

Solution: If matrix is 2×2 let $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$ then eigen value is given by

$$\begin{pmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{pmatrix}=0\Rightarrow(1-\lambda)(4-\lambda)-4=0\Rightarrow\lambda=0,5$$

If matrix is 3×3, let $\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix}$ then eigen value is given by

$$\begin{pmatrix}1-\lambda & 2 & 3\\ 2 & 4-\lambda & 6\\ 3 & 6 & 9-\lambda\end{pmatrix}=0$$

$$(1-\lambda)[(4-\lambda)(9-\lambda)-36]+2[18-2(9-\lambda)]+3[12-3(4-\lambda)]$$

$$(1-\lambda)[\lambda^2-13\lambda+36-36]+2[18-18+2\lambda]+3[12-12+3\lambda]=0$$

$$\lambda^2-13\lambda-\lambda^3+13\lambda^2+13\lambda=0\Rightarrow\lambda^3-14\lambda^2=0\Rightarrow\lambda=0,0,\lambda=14$$

i.e. has one degenerate eigenvalue with degeneracy 2.

Thus one can generalized that for n dimensional matrix has one degenerate eigevalue with degeneracy $(n-1)$.

Ans. 26: (c)

Solution: If A and P be square matrices of the same type and if P be invertible then matrices A and $B = P^{-1}AP$ have the same characteristic roots

Then $B-\lambda I=P^{-1}AP-P^{-1}\lambda IP=P^{-1}(A-\lambda I)P$ where I is identity matrix.

$$|B-\lambda I|=|P^{-1}(A-\lambda I)P|=|P^{-1}||A-\lambda I||P|=|A-\lambda I||P^{-1}||P|=|A-\lambda I||PP^{-1}|=|A-\lambda I|$$

Thus the matrices A and B (= $P^{-1}AP$) have the same characteristic equation and hence characteristic roots of eigen values. Since the sum of the eigen values of a matrix and product of eigen values of a matrix is equal to the determinant of matrix hence third alternative is incorrect.

Ans. 27: (c)

Solution: We know that for any matrix

1. The product of eigenvalues is equal to determinant of that matrix.
2. $\lambda_1+\lambda_2+\lambda_3+.....=$ Trace of matrix

$\lambda_1+\lambda_2+\lambda_3=11$ and $\lambda_1\lambda_2\lambda_3=36$. Hence the largest eigen value of the matrix is 9.

Ans. 28: (b)

Solution: the characteristic equation is given by $\begin{pmatrix}0-\lambda & 1 & 1\\ 0 & 0-\lambda & 1\\ 1 & 0 & 0-\lambda\end{pmatrix}=0$

$$-\lambda\left((-\lambda)^2-0\right)-1(1)+1(0-(-\lambda)=\lambda^3-\lambda-1=0$$

So from calley hemilton's theorem $M^3-M-I=0$ multiply both side by M^{-1}

$M^2-I-M^{-1}=0$ so $M^{-1}=M^2-I$.

MSQ (Multiple Select Questions)

Ans. 29: (a), (b) and (c)

Solution: For eigen value $|A-\lambda I|=0$

$$\begin{vmatrix}5-\lambda & 0 & 2\\ 0 & 1-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix}=0\Rightarrow(5-\lambda)\left[(1-\lambda)(2-\lambda)\right]+2(-2)(1-\lambda)=0$$

$\Rightarrow-\lambda^3+8\lambda^2-13\lambda+6=0$, eigenvalue $\lambda=1,1,6$

eigen vector can be determine by $AX_1=\lambda_1X_1$

For $\lambda=1$,

$$\begin{bmatrix}5 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\Rightarrow 5x_1+2x_3=x_1,\quad x_2=x_2,2x_1+2x_3=x_3\Rightarrow x_2=0,x_3=-2x_1$$

$X_1 = \begin{bmatrix} x_1 \\ 0 \\ -2x_1 \end{bmatrix}$ from orthogonality condition $X_1^T X_1 = 1 \Rightarrow x_1 = \frac{1}{\sqrt{5}} \Rightarrow X_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Hence $\lambda = 1$ is degenerate its eigen vector will $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ -2x_1 \end{bmatrix}$ will orthogonal to X_1 so

$X_1^T X_2 = 0 \Rightarrow 4x_1 = 0$ so $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 6$

$$\begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 6 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow 5x_1 + 2x_3 = 6x_1, \quad x_2 = 6x_2, 2x_1 + 2x_3 = 6x_3 \Rightarrow x_1 = 2x_3, x_2 = 0,$$

$X_3 = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. From orthogonality condition, $x_3 = \frac{1}{\sqrt{5}} \Rightarrow X_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Ans. 30: (a), (b), (c) and (d)

Solution: For eigen value $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[-\lambda(1-\lambda)\right] + \lambda = 0 \Rightarrow -\lambda^3 + 2\lambda^2 = 0$$

eigenvalue $\lambda = 0, 0, 2$

eigen vector can be determine by $AX_1 = \lambda_1 X_1$

For $\lambda = 0$,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_1 + x_2 = 0, \quad x_3 = 0 \Rightarrow x_1 = -x_2$$

$X_1 = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix}$. From orthogonality condition, $X_1^T X_1 = 1 \Rightarrow x_1 = \frac{1}{\sqrt{2}} \Rightarrow X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

One can not find independent vector directly so we can find it from block diagonal method so another eigen vector which is independent to $\Rightarrow X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and also eigenvector corresponds to $\lambda = 0$ is $\Rightarrow X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 2$,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_1 + x_2 = 2x_1, x_3 = 0 \Rightarrow x_1 = x_2 \Rightarrow X_3 = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix}$$

From orthogonality condition $X_3^T X_3 = 1 \Rightarrow x_2 = \frac{1}{\sqrt{2}} \Rightarrow X_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Ans. 31: (a) and (c)

Solution: For eigen value $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 0 & \sqrt{3} \\ 0 & 3-\lambda & 0 \\ \sqrt{3} & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)\left[(3-\lambda)^2 - 0\right] + \sqrt{3}\left[-\sqrt{3}(3-\lambda)\right] = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0 \Rightarrow \lambda = 2, 3, 6$$

eigen vector can be determine by $AX_1 = \lambda_1 X_1$

For $\lambda = 2$,

$$\begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$5x_1 + \sqrt{3}x_3 = 2x_1, \quad 3x_2 = 2x_2, \sqrt{3}x_1 + 3x_3 = 2x_3 \Rightarrow x_2 = 0,\ x_3 = -\sqrt{3}x_1$

$X_1 = \begin{bmatrix} x_1 \\ 0 \\ \sqrt{3}x_1 \end{bmatrix}$. From orthogonality condition $X_1^T X_1 = 1 \Rightarrow x_1 = \frac{1}{2} \Rightarrow X_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -\sqrt{3} \end{bmatrix}$

For $\lambda = 3$,

$$\begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow 5x_1 + \sqrt{3}x_3 = 3x_1, 3x_2 = 3x_2, \sqrt{3}x_1 + 3x_3 = 3x_3 \Rightarrow x_1 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 6$,

$$\begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 6\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow 5x_1 + \sqrt{3}x_3 = 6x_1, 3x_2 = 6x_2 \Rightarrow x_1 = \sqrt{3}x_3,\ x_2 = 0$$

$X_3 = \begin{bmatrix} \sqrt{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$. From orthogonality condition $X_3^T X_3 = 1 \Rightarrow x_3 = \frac{1}{2} \Rightarrow X_3 = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 0 \\ 1 \end{bmatrix}$

Ans. 32: (a) and (d)

Solution: $(AB + BA)^\dagger = B^\dagger A^\dagger + A^\dagger B^\dagger \Rightarrow -BA - AB = -(AB + BA)$

$-i(AB + BA)^\dagger = -i(B^\dagger A^\dagger + A^\dagger B^\dagger) \Rightarrow -i(-BA - AB) = i(AB + BA)$

Ans. 33: (a), (b), (c) and (d)

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(\lambda^2-1)=0 \Rightarrow \lambda^3-\lambda^2-\lambda+1=0$$

$$A^3-A^2-A+I=0 \Rightarrow A^2-A-I+A^{-1}=0$$

$$A^2=I \Rightarrow A^2-A-I+A^{-1}=0 \quad \Rightarrow I-A-I-A^{-1}=0 \Rightarrow A^{-1}=A$$

NAT (Numerical Answer Type)

Ans. 34: 5

$$\begin{bmatrix} 4-\lambda & -1 & -1 \\ -1 & 4-\lambda & -1 \\ -1 & -1 & 4-\lambda \end{bmatrix} \Rightarrow (4-\lambda)\begin{bmatrix} 1 & -1 & -1 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} = (4-\lambda)(5-\lambda)^2=0 \Rightarrow \lambda=2,5,5.$$

Ans35: 0

Solution: $|A-\lambda I|=0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(\lambda^2-1)+\lambda=0 \Rightarrow \lambda=0, +\sqrt{2}, -\sqrt{2}$

Ans. 36: 1

Solution: $A=\begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{bmatrix}=0$

$$\Rightarrow (-\lambda)(1-\lambda)(-\lambda)+(-(1-\lambda)=0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2-1\right)=0 \quad \Rightarrow (1-\lambda)\left(\lambda^2-1\right)=0 \quad \lambda=1,1,-1$$

2. Complex Numbers

2.1 Definition

A number of the form $x+iy$, where x and y are real numbers and $i=\sqrt{(-1)}$ is called complex number. x is called the real part of $x+iy$ and is written as $R(x+iy)$ and y is called the imaginary part and is written as $I(x+iy)$.

A pair of complex numbers $x+iy$ and $x-iy$ are said to be conjugate of each other.

Properties

(1) If $x_1+iy_1=x_2+iy_2$, then $x_1-iy_1=x_2-iy_2$

(2) Two complex numbers $x_1+iy_1=$ and x_2+iy_2 are said to be equal when

$R(x_1+iy_1)=R(x_2+iy_2)$, i.e $x_1=x_2$ and $I(x_1+iy_1)=I(x_2+iy_2)$, i.e., $y_1=y_2$

(3) Sum, difference product and quotient of any of two complex number is itself a complex number. If x_1+iy_1 and x_2+iy_2 be two given complex numbers, then

(i) Their sum $=(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2)$

(ii) Their difference $=(x_1+iy_1)-(x_2+iy_2)=(x_1-x_2)+i(y_1-y_2)$

(iii) Their product $=(x_1+iy_1)\times(x_2+iy_2)=x_1x_2-y_1y_2+i(x_1y_2+x_2y_2)$

(iv) Quotient $=\dfrac{x_1+iy_1}{x_2+iy_2}=\dfrac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}=\dfrac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\dfrac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$

(4) Every complex number $x+iy$ can always be expressed in the form $r(\cos\theta+i\sin\theta)$.

Put $R(x+iy)$, i.e. $x=r\cos\theta$ (i)

and $I(x+iy)$ i.e. $y=r\sin\theta$ (ii)

Squaring and adding, we get $x^2+y^2=r^2$ i.e. $r=\sqrt{(x^2+y^2)}$ (taking positive square root only)

Dividing (ii) by (i), we get $\dfrac{y}{x}=\tan\theta$ i.e. $\theta=\tan^{-1}\left(\dfrac{y}{x}\right)$

Thus $x+iy=r(\cos\theta+i\sin\theta)$ where $r=\sqrt{(x^2+y^2)}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

The number $r=+\sqrt{(x^2+y^2)}$ is called the modulus of $x+iy$ and is written as mod $|x+iy|$.

The angle θ is called the amplitude or argument of $x+iy$ and is written as amp $(x+iy)$ or $\arg(x+iy)$. Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the principal value of the amplitude. Unless otherwise specified, we shall take $amp(z)$ to mean the principal value.

Note: $\cos\theta+i\sin\theta$ is briefly written as $cis\theta$ (pronounced as $sis\theta$)

(5) If the conjugate of $z=x+iy$ be $\bar{z}$, then

(i) $R(z)=\frac{1}{2}(z+\bar{z}), I(z)=\frac{1}{2i}=(z-\bar{z})$

(ii) $|z|=\sqrt{R^2(z)+I^2(z)}=|\bar{z}|$

(iii) $z\bar{z}=|z|^2$

(iv) $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$

(v) $\overline{z_1z_2}=\bar{z}_1.\bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$ where $\bar{z}_2\neq 0$

Example: Reduce $1-\cos\alpha+i\sin\alpha$ to the modulus amplitude form.

Solution: Put $1-\cos\alpha=r\cos\theta$ and $\sin\alpha=r\sin\theta$

$$\therefore r^2=(1-\cos\alpha)^2+\sin^2\alpha=2-2\cos\alpha=4\sin^2\frac{\alpha}{2}$$

$$\Rightarrow r=2\sin\frac{\alpha}{2} \text{ and } \tan\theta=\frac{\sin\alpha}{1-\cos\alpha}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}=\cot\frac{\alpha}{2}=\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\Rightarrow\theta=\frac{\pi-\alpha}{2}$$

$$\Rightarrow 1-\cos\alpha+i\sin\alpha=2\sin\frac{\alpha}{2}\left[\cos\left(\frac{\pi-\alpha}{2}\right)+i\sin\left(\frac{\pi-\alpha}{2}\right)\right]$$

$$\Rightarrow 1-\cos\alpha+i\sin\alpha=2\sin\frac{\alpha}{2}\left[\sin\left(\frac{\alpha}{2}\right)+i\cos\left(\frac{\alpha}{2}\right)\right]$$

Example: Find the complex number z if $\arg(z+1)=\frac{\pi}{6}$ and $\arg(z-1)=\frac{2\pi}{3}$

Solution: Let $z=x+iy$ so that $z+1=(x+1)+iy$ and $(z-1)=(x-1)+iy$

$\arg(z+1)=\frac{\pi}{6},\ \therefore \tan^{-1}\left(\frac{y}{x+1}\right)=30^0$

i.e. $\frac{y}{x+1}=\tan 30^0=\frac{1}{\sqrt{3}}\Rightarrow\sqrt{3y}=x+1$ (i)

Also since $\arg(z-1)=\frac{2\pi}{3}\ \therefore \tan^{-1}\left(\frac{y}{x-1}\right)=120^0$

i.e $\frac{y}{x-1}=\tan 120^0=-\sqrt{3}\ \Rightarrow y=-\sqrt{3x}+\sqrt{3}$

$\Rightarrow\sqrt{3y}=-3x+3$ (ii)

From above equations; $4x-2=0\Rightarrow x=\frac{1}{2}\Rightarrow\sqrt{3}y=\frac{1}{2}+1\Rightarrow y=\frac{\sqrt{3}}{2}\ \Rightarrow z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Example: Find the real values of x, y so that $-3+ix^2y$ and x^2+y+4i may represent complex conjugate numbers.

Solution: If $z=-3+ix^2y$, then $\bar{z}=x^2+y+4i$

So that $z=(x^2+y)-4i$

$\therefore\quad -3+ix^2y=x^2+y-4i$

Equating real and imaginary parts from both sides. we get

$-3=x^2+y, x^2y=-4$

Eliminating $x, (y+3)y=-4$

or $y^2+3y-4=0$ i.e. $y=1$ or -4

when $y=1,\quad x^2=-3-1$ or $x+2i$ which is not feasible

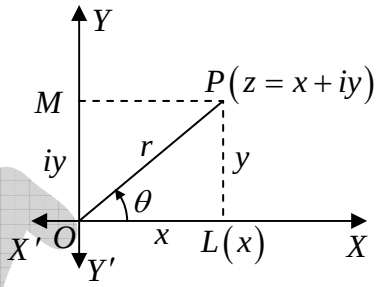
when $y=-4,\quad x^2=1$ or $x=\pm1$

Hence $x=1,\quad y-4$ or $x=-1,\ y=-4$

2.2 Geometric representation of complex numbers

Consider two lines $X'OX$, $Y'OY$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the real axis) positive real numbers being along OX and negative ones along OX'. Let the point L on OX represent the real number x.

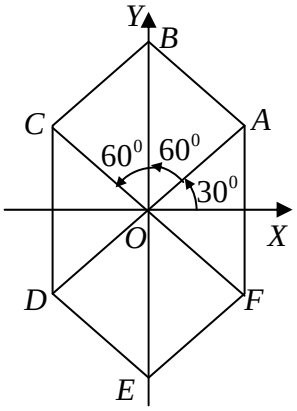


Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'OY$ (called the imaginary axis) the positive ones along OY and negative ones along OY'. Let the point M on OY represent the imaginary number iy.

Complete the rectangle *OLPM*. Then the point whose Cartesian coordinates are (x, y) uniquely represents the complex number $z = x + iy$ on the complex plane z. The diagram in which this representation is carried out is called the Argand's diagram.

If (r, θ) be the polar coordinates of P, then r is the modulus of z and θ is its amplitude.

Example: The centre of a regular hexagon is a origin and on vertex is given by $\sqrt{3}+i$ on the Argand diagram. Determine the other vertices.



Solution: Let $\vec{OA} = \sqrt{3}+i$ so that

$$OA = 2 \text{ and } \angle XOA = \tan^{-1} 1/\sqrt{3} = 30^0$$

Being a regular hexagon, $OB = OC = 2$

$$\angle XOB = 30^0 + 60^0 = 90^0$$

and $$\angle XOC = 30^0 + 120^0 = 150^0$$

$\therefore$ $$\vec{OB} = 2\left(\cos 90^0 + i \sin 90^0\right) = 2i$$

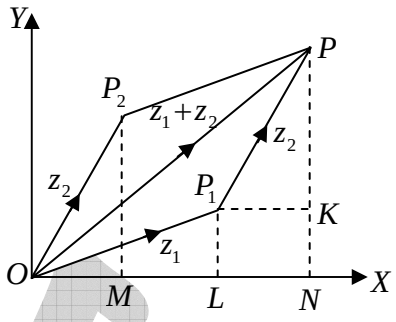
$$\vec{OC} = 2\left(\cos 150^0 + i \sin 150^0\right) = -\sqrt{3} + i$$

since $\overrightarrow{AD}, \overrightarrow{BE}, \overrightarrow{CF}$ are bisected at O, $\therefore$ $\overrightarrow{OD} = -\overrightarrow{OA} = -\sqrt{3} - i$

$\overrightarrow{OE} = -\overrightarrow{OB} = -2i$ and $\overrightarrow{OF} = -\overrightarrow{OC} = \sqrt{3} - i$.

Geometric Representation of $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Complete the parallelogram OP_1PP_2. Draw P_1L, P_2M and $PN \perp s$ to OX. Also draw $P_1K \perp PN$.



Since

$ON = OL + LN = OL + OM = x_1 + x_2 \quad [\because LN = P_1K = OM]$

and $\quad NP = NK + KP = LP_1 + MP_2 = y_1 + y_2$

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2$$

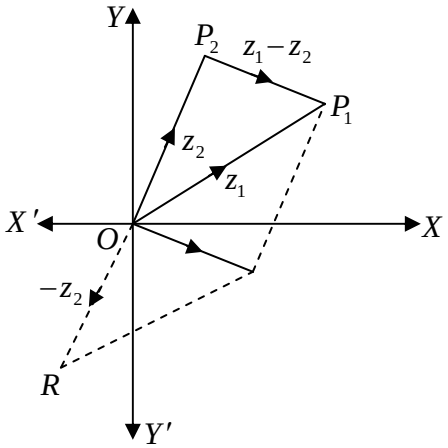
Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and } \text{amp}(z_1 + z_2) = \angle XOP$$

Corollary, we have $\overrightarrow{OP_1} + \overrightarrow{P_1P} = \overrightarrow{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.



Produce P_2O backwards to R such that $OR = OP_2$. Then coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$

Complete the parallelogram $ORQP_1$. Then sum of z_1 and $(-z_2)$ is represented by OQ i.e.,

$$z_1 - z_2 = \overrightarrow{OQ} = \overrightarrow{P_2P_1}$$

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1

Corollary, By means of the relation $\overrightarrow{P_2P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_2}$, any vector $\overrightarrow{P_2P_1}$ may be referred to the origin.

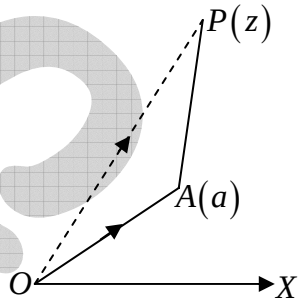
Example: Find the locus of $P(z)$ when

(i) $|z-a| = k$;

(ii) amp $(z-a) = \alpha$, where k and α are constants

Solution: Let a, z be represented by A and P in the complex plane, O being the origin.

Then $z - a = \overrightarrow{OP} - \overrightarrow{OA} = \overrightarrow{AP}$



(i) $|z-a| = k$ means that $AP = k$

Thus the locus of $P(z)$ is a circle whose centre is $A(\alpha)$ and radius k.

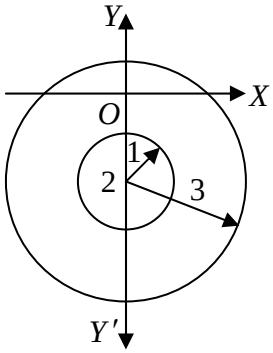
(ii) amp $(z-a)$, i.e., amp$\left(\overrightarrow{AP}\right) = \alpha$, means that AP always makes a constant angle with the X -axis.

Thus the locus of $P(z)$ is a straight line through $A(\alpha)$ making an $\angle\alpha$ with OX.

Example: Determine the region in the z -plane represented by

(i) $1 < |z+2i| \le 3$ (ii) $R(z) > 3$ (iii) $\frac{\pi}{6} \le amp(z) \le \frac{\pi}{3}$

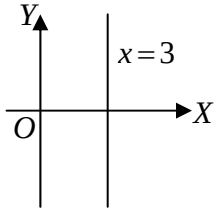
Solution: (i) $|z+2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z+2i| = 3$ is a circle with the same centre and radius 3.



Hence $1 < |z+2i| \le 3$ represents the region outside the circle $|z+2i| = 1$ and inside (including circumference of) the circle $|z+2i| = 3$.

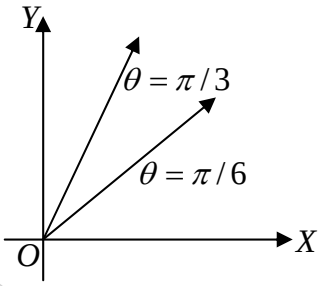
(ii) $R(z) > 3$, define all points (z) whose real part is greater than 3.

Hence it represents the region of the complex plane to the right of the line $x = 3$.



(iii) If $z = r(\cos\theta + i\sin\theta)$, then $\text{amp}(z) = \theta$

$\therefore \frac{\pi}{6} \le amp(z) \le \frac{\pi}{3}$ defines the region bounded by and including the lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{3}$

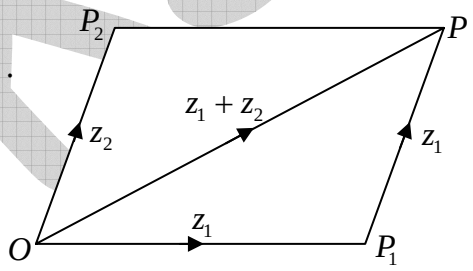


Example: If z_1, z_2 be any two complex numbers prove that

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ i.e., [the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli]

(ii) $|z_1 - z_2| \ge |z_1| - |z_2|$ i.e., [the modulus of the difference of two complex number is greater than or at the most equal to the difference of their moduli].

Solution: (i) Let P_1, P_2 represent the complex numbers z_1, z_2.



Complete the parallelogram OP_1PP_2, so that

$|z_1| = OP_1, |z_2| OP_2 = P_1P$,

and $|z_1 + z_2| = OP$

Now from $\Delta OP_1P, OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

Hence $|z_1 + z_2| \le |z_1| + |z_2|$ (i)

Again $|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2|$

Thus $|z_1 - z_2| \ge |z_1| - |z_2|$ (ii)

Corollary, $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

In general, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$.

Example: If $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of amplitude of z_1 and z_2 is $\frac{\pi}{2}$.

Solution: Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then $2z_1 = r\left[\left(\cos\theta+\cos\phi\right)+i\left(\sin\theta+\sin\phi\right)\right]$

$= r\left\{2\cos\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2}+2i\sin\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2}\right\}$

or $\quad z_1 = r\cos\dfrac{\theta-\phi}{2}\left(\cos\dfrac{\theta+\phi}{2}+i\sin\dfrac{\theta+\phi}{2}\right)$ i.e., $\text{amp}\left(z_1\right)=\dfrac{\theta+\phi}{2}$ (i)

Also $\quad 2z_2 = r\left(\cos\theta-\cos\phi\right)+i\left(\sin\theta-\sin\phi\right) = 2r\sin\dfrac{\theta-\phi}{2}\left(-\sin\dfrac{\theta+\phi}{2}+i\cos\dfrac{\theta+\phi}{2}\right)$

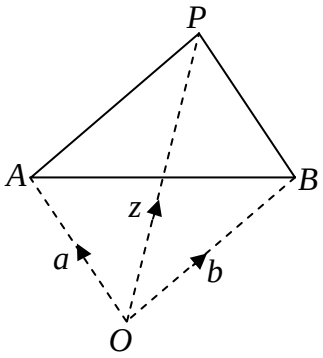
or $\quad z_2 = r\sin\dfrac{\theta-\phi}{2}\left\{\cos\left(\dfrac{\pi}{2}+\dfrac{\theta+\phi}{2}\right)+i\sin\left(\dfrac{\pi}{2}+\dfrac{\theta+\phi}{2}\right)\right\}$

i.e., $\quad \text{amp}\left(z_2\right)=\dfrac{\pi}{2}+\dfrac{\theta+\phi}{2}$ (ii)

Hence [(ii) – (i), gives $\text{amp}\left(z_2\right)-\text{amp}\left(z_1\right)=\dfrac{\pi}{2}$

Example: Find the locus of the point z, when $\left|\dfrac{z-a}{z-b}\right|=k$

Solution: Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point.



(i) Since $|z-a| = AP$ and $|z-b| = BP$

$\therefore$ The point P moves so that $\left|\dfrac{z-a}{z-b}\right| = \left|\dfrac{z-a}{z-b}\right| = \dfrac{AP}{BP} = k$

i.e., P moves so that its distance from two fixed points are in a constant ratio, which is obviously the Appollonius circle.

When $k=1, BP = AP$ i.e., P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB

Hence the locus of $P(z)$ is a circle (unless $k=1$, when the locus is the right bisector of AB)

Example: If $z_1 z_2$ be two complex numbers, show that $(z_1+z_2)^2+(z_1-z_2)^2=2\left[|z_1|^2+|z_2|^2\right]$

Solution: Let $z_1=r_1\,\mathrm{c}i\mathrm{s}\theta_1$ and $z_2=r_2 cis\theta_2$ so that

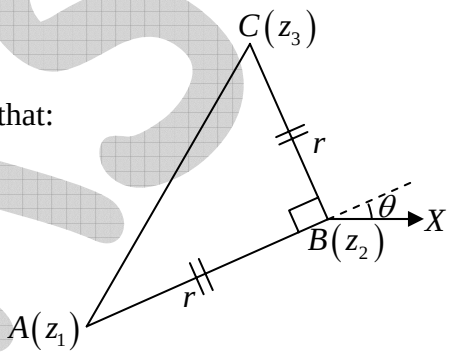
$$|z_1+z_2|^2=(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2=r_1^2+r_2^2+2r_1r_2\cos(\theta_2-\theta_1)$$

$$|z_1-z_2|^2=(r_1\cos\theta_1-r_2\cos\theta_2)^2+(r_1\sin\theta_1-r_2\sin\theta_2)^2=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)$$

$$\therefore\ |z_1+z_2|^2+|z_1-z_2|^2=2(r_1^2+r_2^2)=2\left\{|z_1|^2|z_2|^2\right\}$$

Example: If z_1,z_2,z_3 be the vertices of an isosceles triangle, right angled at z_2 prove that

$$z_1^2+z_3^2+2z_2^2=2z_2(z_1+z_3)$$



Solution: Let $A(z_1),B(z_2),C(z_3)$ be the vertices of ΔABC such that:

$AB=BC$ and $\angle ABC=\dfrac{\pi}{2}$. Then $|z_1-z_2|=|z_3-z_2|=r$ (say)

If $\mathrm{amp}(z_1-z_2)=\theta$ then $\mathrm{amp}(z_3-z_2)=\dfrac{\pi}{2}+\theta$

$$\therefore\quad z_1-z_2=r(\cos\theta+i\sin\theta),$$

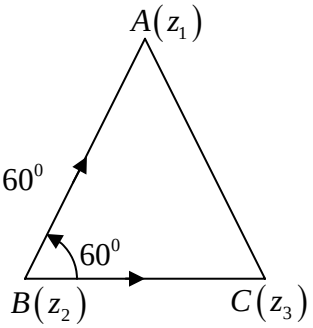
$$z_3-z_2=r\left[\cos\left(\frac{\pi}{2}+\theta\right)+i\sin\left(\frac{\pi}{2}+\theta\right)\right]=r(-\sin\theta+i\cos\theta)$$

$$z_3-z_2=ir(\cos\theta+i\sin\theta)=i(z_1-z_2)$$

$$(z_3-z_2)^2=-(z_1-z_2)^2 \text{ or } z_1^2+z_3^2+2z_2^2=2z_3(z_1+z_2).$$

Example: If z_1,z_2,z_3 be the vertices of an equilateral triangle, prove that

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$



Solution: Since ΔABC is equilateral, therefore, BC when rotated through 60^0 coincides with BA. But to turn the direction of a complex number through an $\angle\theta$, we multiply it by $\theta+i\sin\theta$

$$\therefore\ \overrightarrow{BC}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\overrightarrow{BA}\ \Rightarrow(z_3-z_2)\left(\frac{1+i\sqrt{3}}{2}\right)=z_1-z_2\ \Rightarrow i\sqrt{3}(z_3-z_2)=2z_1-z_2-z_3$$

$$\Rightarrow -3(z_3-z_2)^2=(2z_1-z_2-z_3)^2\ \Rightarrow 4(z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1)=0$$

2.3 De Moivre's Theorem

Statement: If n be

(i) an integer, positive or negative $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$;

(ii) a fraction, positive or negative, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$

Proof: (i) When n is a positive integer.

By actual multiplication

$cis\theta_1 cis\theta_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$ i.e. $cis(\theta_1+\theta_2)$

Similarly $cis\theta_1 cis\theta_2 cis\theta_3 = cis(\theta_1+\theta_2)cis\theta_3 = cis(\theta_1+\theta_2+\theta_3)$

Proceeding in this way

$cis\theta_1 cis\theta_2 cis\theta_3 ... cis\theta_n = cis(\theta_1+\theta_2+\theta_3+...+\theta_n)$

Now putting $\theta_1 = \theta_2 = \theta_3 = ...\theta_n = \theta,$ we obtain $(cis\theta)^n = cisn\theta$

(ii) When n is a negative integer.

Let $n = -m$, where m is a $+ve$ integer.

$$(cis\theta)^n = (cis\theta)^{-m} = \frac{1}{cism\theta} = \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m\theta) + i\sin(-m\theta) = cis(-m\theta) = cisn\theta \qquad (\because\ -m = n)$$

Example: Simplify $\dfrac{(\cos 3\theta + i\sin\theta)^4 (\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3 (\cos 5\theta + + i\sin 5\theta)^{-4}}$

Solution: We have $(\cos 3\theta + i\sin 3\theta)^4 = \cos 12\theta + i\sin 12\theta = (\cos\theta + i\sin\theta)^{12}$

$(\cos 4\theta - i\sin 4\theta)^5 = (\cos 20\theta - i\sin 20\theta) = (\cos\theta + i\sin\theta)^{-20}$

$(\cos 4\theta + i\sin 4\theta)^3 = \cos 12\theta + i\sin 12\theta = (\cos\theta + i\sin\theta)^{12}$

$(\cos 5\theta + i\sin 5\theta)^{-4} = \cos 20\theta - i\sin 20\theta = (\cos\theta + i\sin\theta)^{-20}$

The given expression $= \dfrac{(\cos\theta + i\sin\theta)^{12}(\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{12}(\cos\theta + i\sin\theta)^{-20}} = 1$

Example: Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\dfrac{\theta}{2}\right).\left(\cos\dfrac{n\theta}{2}\right)$

Solution: $1+\cos\theta = r\cos\alpha, \quad \sin\theta = r\sin\alpha$

$r^2 = (1+\cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\dfrac{\theta}{2} \Rightarrow r = 2\cos\dfrac{\theta}{2}$

$\tan\alpha = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\dfrac{\theta}{2} \Rightarrow \alpha = \dfrac{\theta}{2}$

$\left[r(\cos\alpha + i\sin\alpha)\right]^n + \left[r(\cos\alpha - i\sin\alpha)\right]^n = r^n\left[(\cos\alpha + i\sin\alpha)^n + (\cos\alpha - i\sin\alpha)^n\right]$

$= r^n(\cos n\alpha + i\sin n\alpha + \cos n\alpha - i\sin n\alpha) = r^2.2\cos n\alpha$ [Substituting the values of r and α]

$= 2^{n+1}\cos^n\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{n\theta}{2}\right)$

Example: If $2\cos\theta = x + \dfrac{1}{x}$, prove that

(i) $\cos r\theta = x^2 + \dfrac{1}{x^r}2,$ (ii) $\dfrac{x^{2n}+1}{x^{2n-1}+x} = \dfrac{\cos n\theta}{\cos(n-1)\theta}$

Solution: Since $x + \dfrac{1}{x} = 2\cos\theta \quad \therefore x^2 - 2x\cos\theta + 1 = 0$

Hence $x = \dfrac{2\cos\theta \pm \sqrt{(4\cos^2\theta - 4)}}{2} = \cos\theta \pm i\sin\theta$

(i) Taking the $+ve$ sign, $x^r = (\cos\theta + i\sin\theta)^r = \cos r\theta + i\sin r\theta$

and $x^{-r} = (\cos\theta + i\sin\theta)^{-r} = \cos r\theta - i\sin r\theta$

Adding $x^r + \dfrac{1}{x^r} = 2\cos r\theta$.

Similarly with the $-ve$ sign the same result follows

(ii) $\dfrac{x^{2n}+1}{x^{2n-1}+x}=\dfrac{(\cos\theta+i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}=\dfrac{\cos 2n\theta+i\sin 2n\theta+1}{\cos(2n-1)\theta+i\sin(2n-1)\theta+\cos\theta+i\sin\theta}$

$=\dfrac{(1+\cos 2n\theta)+i\sin 2n\theta}{(\cos(2n-1)\theta+\cos\theta)+i(\sin(2n-1)\theta+\sin\theta)}=\dfrac{2\cos^2 n\theta+2i\sin n\theta\cos n\theta}{2\cos n\theta\cos(n-1)\,\theta+2i\sin n\theta\cos(n-1)\,\theta}$

$=\dfrac{\cos n\theta(2\cos n\theta+2i\sin n\theta)}{\cos(n-1)\theta(2\cos n\theta+2i\sin n\theta)}=\dfrac{\cos n\theta}{\cos(n-1)\theta}$

Example: If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$

Prove that

(i) $\sin 2\alpha+\sin 2\beta+\sin\gamma=0$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$

(iv) $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$

Solution: Let $a=cis\alpha$, $b=cis\beta$ and $c=cis\gamma$

Then $a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$

$\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=(\cos\alpha+\cos\beta+\cos\gamma)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$

$=\sum\dfrac{\cos\alpha-i\sin\alpha}{\cos\alpha-i\sin\alpha}.\dfrac{1}{\cos\alpha+i\sin\alpha}=\sum(\cos\alpha-i\sin\alpha)$

$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0$

or $bc+ca+ab=0$

$\therefore\quad a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=0$

or $(cis\alpha)^2+(cis\beta)^2+(cis\gamma)^2=cis2\alpha+cis2\beta+cis2\gamma=0$

Equating imaginary parts from both sides, we get $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$

(ii) Since $a+b+c=0$, $\therefore \quad a^3+b^3+c^3=3abc$

$(cis\alpha)^3+(cis\beta)^3+(cis\gamma)^3=3cis\alpha\, cis\beta\, cis\gamma$

$cis3\alpha+cis3\beta+cis3\gamma=3cis(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $a+b=-c$ or $(a+b)^2=c^2$ or $a^2+b^2-c^2=-2ab$

Again squaring, $a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$

i.e., $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $(cis4\alpha)^4+(cis\beta)^4+(cis\gamma)^4=2\sum(\cos\alpha)^2(cis\beta)^2$

or $cis4\alpha+cis4\beta+cis4\gamma=2\sum cis2\alpha cis2\beta=2\sum cis2(\alpha+\beta)$

Equating imaginary parts from both sides, we get $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$

(iv) $ab+bc+ca=0$ or $cis\alpha cis\beta+cis\beta cis\gamma+cis\gamma cis\alpha=0$

or $cis(\alpha+\beta)+cis(\beta+\gamma)+cis(\gamma+\alpha)=0$

Equating imaginary parts from both sides, we get $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$

Example: Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.

Solution: If x be a cube root of unity, then

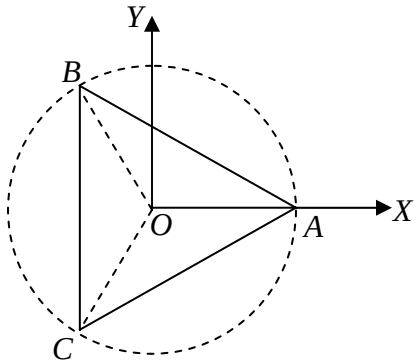
$$x=(1)^{\frac{1}{3}}=(\cos 0+i\sin 0)^{\frac{1}{3}}=(cis0)^{\frac{1}{3}}=cis(2n\pi)^{\frac{1}{3}}=cis\frac{2n\pi}{3}$$

where $n=0,1,2$

$\therefore$ the three values of x are $cis0=1$

$$cis\frac{2\pi}{3}=\cos 120^0+i\sin 120^0=-\frac{1}{2}+i\frac{\sqrt{3}}{2},$$

and $$cis\frac{4\pi}{3}=\cos 240^0+i\sin 240^0=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$



These three cube roots are represented by the points, A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120^0, \angle AOC = 240^0$.

$\therefore$ These points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120^0$ i.e., $AB = BC = CA$

Hence A, B, C form an equilateral triangle.

Example: Find all the values of $\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^{\frac{3}{4}}$. Also show that the continued product of these values is 1.

Solution: Put $\frac{1}{2} = r\cos\theta$ and $\frac{\sqrt{3}}{2} = r\sin\theta$ so that $r = 1$ $\theta = \frac{\pi}{3}$

$\therefore$ $\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^{\frac{3}{4}} = \left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}} = \left(cis\pi\right)^{\frac{1}{4}}$

$= \left[cis(2n+1)\pi\right]^{\frac{1}{4}} = cis(2n+1)\frac{\pi}{4}$ where $n = 0,1,2,3$

Hence the required values are $cis\frac{\pi}{4}, cis\frac{3\pi}{4}, cis\frac{5\pi}{4}$ and $cis\frac{7\pi}{4}$.

$\therefore$ Their continued product $= cis\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right) = cis4\pi = 1$

Example: Use De Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

Solution: $x^4 - x^3 + x^2 - x + 1$ is a G.P. with common ratio $(-x)$

$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1$ or $x^5 + 1 = 0$

$x^5 = -1 = cis\pi = cis(2n+1)\pi$

$\therefore\ x = \left[cis(2n+1)\pi\right]^{\frac{1}{5}} = cis(2n+1)\frac{\pi}{5}$, where $n = 0,1,2,3,4$

Hence the values are $cis\frac{\pi}{5}, cis\frac{3\pi}{5}, cis\pi, cis\frac{7\pi}{5}, cis\frac{9\pi}{5}$

$$\cos\frac{\pi}{5}+i\sin\frac{\pi}{5},\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5},-1,\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5},\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x+1$, the required roots are:

$$\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$$

Example: Show that the roots of the equation $(x-1)^n=x^n, n$ being a positive integer are $\frac{1}{2}\left(1+i\cot\frac{r\pi}{n}\right)$ where r has values $1,2,3,...,n-1$.

Solution: Given equation is $\left(\frac{x-1}{x}\right)^n=1$ or $1-\frac{1}{x}=(1)^{\frac{1}{n}}$

or $\frac{1}{x}=1-(1)^{\frac{1}{n}}=1-cis\frac{2r\pi}{n}, r=0,1,2,...(n-1)$

or $\frac{1}{x}=\left(1-\cos\frac{2r\pi}{n}\right)-i\sin\frac{2r\pi}{n}=2\sin^2\frac{r\pi}{n}-2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$

$$x=\frac{1}{2\sin\frac{r\pi}{n}}.\frac{1}{\left(\sin\frac{r\pi}{n}-i\cos\frac{r\pi}{n}\right)}=\frac{\sin\frac{r\pi}{n}+i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}=\frac{1}{2}\left(1+i\cot\frac{r\pi}{n}\right), r=1,2,...(n-1)$$

Hence the roots of the given equation are $\frac{1}{2}\left(1+i\cot\frac{r\pi}{n}\right) r=1,2,3...(n-1)$

Example: Find the 7^{th} roots of unity and prove that the sum of their n^{th} power always vanishes unless n be a multiple number of $7, n$ being an integer, and then the sum is 7.

Solution: we have $(1)^{\frac{1}{7}}=(co2r\pi+i\sin 2r\pi)^{\frac{1}{7}}=cis\frac{2r\pi}{7}=cis\left(\frac{2\pi}{7}\right)^r$

Putting $r=0,1,3,4,5,6$, we find that 7^{th} roots of unity are $1,\rho,\rho^2,\rho^3,\rho^4,\rho^5,\rho^6$ where $\rho=\cos\frac{2\pi}{7}$

$\therefore$ Sum S of the n^{th} powers of these roots $= 1+\rho^n+\rho 2n+\ldots+\rho^{6n} = \dfrac{1-\rho^{7n}}{1-\rho^n}$, being a G.P. with common ratio ρ

when n is not a multiple of $7, \rho^{7n} = \left(\rho^7\right)^n = \left(cis2\pi\right)^n = 1$

$1-\rho^{7n} = 0$ and $1-\rho^n \neq 0$, as n is not a multiple of 7. Thus $S = 0$

When n is a multiple of $7 = 7p$ (say)

$$S = 1+\left(\rho^7\right)^p+\left(\rho^7\right)^{2p}+\ldots+\left(\rho^7\right)^{6p} = 1+1+1+1+1+1+1 = 7$$

Example: Find the equation whose roots are $2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$

Solution: Let $y = \cos\theta + i\sin\theta$, where $\theta = \dfrac{\pi}{7}, \dfrac{3\pi}{7}, \ldots\dfrac{13\pi}{7}$

Then $y^7 = \left(\cos\theta+i\sin\theta\right)^7 = \cos 7\theta + i\sin 7\theta = -1$ or $y^7+1=0$

or $\left(y+1\right)\left(y^6-y^5+y^4-y^3+y^2-y+1\right)=0$

Leaving the factor $y+1$ which corresponds to $\theta=\pi$,

We get $y^6-y^5+y^4-y^3+y^2-y+1=0$ (i)

Its roots are $y = cis\theta$ where $\theta = \dfrac{\pi}{7}, \dfrac{3\pi}{7}, \dfrac{5\pi}{7}, \dfrac{9\pi}{7}, \dfrac{11\pi}{7}, \dfrac{13\pi}{7}$

Dividing (i) by y^3; $\left(y^3+\dfrac{1}{y^3}\right)-\left(y^2+\dfrac{1}{y^2}\right)+\left(y+\dfrac{1}{y}\right)-1=0$

or $\left\{\left(y+\dfrac{1}{y}\right)^3-3\left(y+\dfrac{1}{y}\right)\right\}-\left\{\left(y+\dfrac{1}{y}\right)^2-2\right\}\left(y+\dfrac{1}{y}\right)-1=0$

or $x^3-x^2-2x+1=0$ where $x = y+\dfrac{1}{y} = 2\cos\theta$

Now since $\cos\dfrac{13\pi}{7} = \cos\dfrac{\pi}{7}, \cos\dfrac{11\pi}{7} = \cos\dfrac{3\pi}{7}, \cos\dfrac{9\pi}{7} = \cos\dfrac{5\pi}{7}$

Hence the roots are $2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$

To Expand $\sin n\theta, \cos n\theta$ **and** $\tan n\theta$ **in powers of** $\sin\theta, \cos\theta$ **and** $\tan\theta$ **respectively**

(*n* being a positive integer)

We have $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta(i\sin\theta) + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta(i\sin\theta)^3 + \ldots$

$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots) + i({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$

Equating real and imaginary parts h both sides, we get

$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$ (i)

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots$ (ii)

Replacing every $\sin^2\theta$ by $1-\cos^2\theta$ in (i) and every $\cos^2\theta$ by $1-\sin^2\theta$ in (ii), we get the desire expansions of $\cos n\theta$ and $\sin n\theta$

Dividing (ii) by (i)

$$\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^2\theta - \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

and dividing numerator and denominator by $\cos^n\theta$.

We get $\tan n\theta = \dfrac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots}$

Example: Express $\cos 6\theta$ in terms of $\cos\theta$.

Solution: we know that $\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$

Put $n=6$, then $\cos 6\theta = \cos^6\theta - {}^6C_2\cos^4\theta\sin^2\theta + {}^6C_4\cos^2\theta\sin^4\theta - {}^6C_6\sin^6\theta$

$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

Addition formulae for any number of angles

We have, $\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

Now $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1+i\tan\theta_1), \cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1+i\tan\theta_2)$ and so on

$$\cos(\theta_1+\theta_2+...+\theta_n)+i\sin(\theta_1+\theta_2+...\theta_n)$$
$$=\cos\theta_1\cos\theta_2...\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)...(1+i\tan\theta_n)=$$
$$=\cos\theta_1\cos\theta_2...\cos\theta_n[1+i(\tan\theta_1+\tan\theta_2+..+\tan\theta_n)+i^2(\tan\theta_1\tan\theta_2+\tan\theta_3+...)$$
$$+i^3(\tan\theta_1\tan\theta_2\tan\theta_3+...)+...+...]$$
$$=\cos\theta_1\cos\theta_2...\cos\theta_n(1+is_1-s_2-is_3+s_4+...)$$

where $s_1=\tan\theta_1+\tan\theta_2+...+\tan\theta_n, s_2=\sum\tan\theta_1\tan\theta_2, s_3=\sum\tan\theta_1 tan\theta_2\tan\theta_3$ etc.

Equating real and imaginary parts, we have

$$\cos(\theta_1+\theta_2+..+\theta_n)=\cos\theta_1\cos\theta_2...\cos\theta_n(1-s_2+s_4-...)$$
$$\sin(\theta_1+\theta_2+...+\theta_n)=\cos\theta_1\cos\theta_2...\cos\theta_n(s_1-s_3+s_5-...)$$

and by division, we get $\tan(\theta_1+\theta_2+...+\theta_3)=\dfrac{s_1-s_3+s_5-...}{1-s_2+s_4-s_6+...}$

Example: If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$, show that $xy+yz+zx=1$

Solution: Let $\tan^{-1}x=\alpha,\tan^{-1}y=\beta,\tan^{-1}z=\gamma$ so that $x=\tan\alpha, y\tan\beta, z=\tan\gamma$

We know that $\tan(\alpha+\beta+\gamma)=\dfrac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}$

$\tan\dfrac{\pi}{2}=\dfrac{x+y+z-xyz}{1-xy-yz-zx}\Rightarrow 1-xy-yz-zx=0\Rightarrow xy+yz+zx=1.$

Example: If $\theta_1,\theta_2,\theta_3$ be three values of θ which satisfy the equation $\tan 2\theta=\lambda\tan(\theta+\alpha)$ and no two of the them differ by a multiple of π, show that $\theta_1+\theta_2+\theta_3+\alpha$ is a multiple of π.

Solution: Given equation can be written as $\dfrac{2t}{1-t^2}=\lambda\dfrac{t+\tan\alpha}{1-t\tan\alpha}$ where $t=\tan\theta$

or $\lambda t^3+(\lambda-2)\tan\alpha.t^2+(2-\lambda)t-\lambda\tan\alpha=0$

$\therefore$ $\tan\theta_1,\tan\theta_2,\tan\theta_3$, being its roots, we have

$s_1=\sum\tan\theta_1=-\dfrac{\lambda-2}{\lambda}\tan\alpha$, $s_2=\sum\tan\theta_1\tan\theta_2=\dfrac{2-\lambda}{\lambda}$ and $s_3=\tan\alpha$

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{s_1-s_3}{1-s_2}=\frac{\left(-1+\frac{2}{\lambda}\right)\tan\alpha-\tan\alpha}{1-\left(\frac{2}{\lambda}-1\right)}=-\tan\alpha=\tan(n\pi-\alpha)$$

Thus $\theta_1+\theta_2+\theta_3=n\pi-\alpha$, whence follows the result

To expand $\sin^m\theta,\cos^n\theta$ or $\sin^m\theta\cos^n\theta$ in a series of sines or cosines of multiples of θ.

If $z=\cos\theta+i\sin\theta$ then $\frac{1}{z}=\cos\theta-i\sin\theta$

By De Moivre's theorem, $z^p=\cos p\theta+i\sin p\theta,\ \frac{1}{z^p}=\cos p\theta-i\sin p\theta$

$$\therefore\qquad z+\frac{1}{z}=2\cos\theta,\ z-\frac{1}{z}=2i\sin\theta;\ z^p+\frac{1}{z^p}=2\cos p\theta,\ z^p-\frac{1}{z^p}=2i\sin p\theta$$

These results are used to expand the powers of $\sin\theta$ or $\cos\theta$ or their products in a series of sines or cosines of multiples of θ.

Example: Expand $\cos^8\theta$ in a series cosines of multiples of θ.

Solution: Let $z=\cos\theta+i\sin\theta$, so that $z+\frac{1}{z}=2\cos\theta$, $z^p+\frac{1}{z^p}=2\cos p\theta$

$$\therefore (2\cos\theta)^8=\left(z+\frac{1}{z}\right)^8$$

$$=z^8+{}^8C_1z^7.\frac{1}{z}+{}^8C_2z^6.\frac{1}{z^2}+{}^8C_3z^5.\frac{1}{z^3}+{}^8C_4z^4.\frac{1}{z^4}+{}^8C_5z^3.\frac{1}{z^5}+{}^8C_6z^2.\frac{1}{z^6}+{}^8C_7z.\frac{1}{z^7}+\frac{1}{z^8}$$

$$=\left(z^8+\frac{1}{z^8}\right)+{}^8C_1\left(z^6+\frac{1}{z^6}\right)+{}^8C_2\left(z^4+\frac{1}{z^4}\right)+{}^8C_3\left(z^2+\frac{1}{z^2}\right)+{}^8C_4$$

$$=(2\cos8\theta)+8(2\cos6\theta)+28(2\cos4\theta)+56(2\cos2\theta)+70$$

Hence $\cos^8\theta=\frac{1}{128}[\cos8\theta+8\cos6\theta+28\cos4\theta+56\cos2\theta+35]$

Example: Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of θ

Solution: Let $z=\cos\theta+i\sin\theta$

So that $z+\frac{1}{z}=2\cos\theta, z-\frac{1}{z}=2i\sin\theta$ and $z^p-\frac{1}{z^p}=2i\sin p\theta$

$$\therefore (2i\sin\theta)^7(2\cos\theta)^3=\left(z-\frac{1}{z}\right)^7\left(z+\frac{1}{z}\right)^3=\left(z-\frac{1}{z}\right)^4\left[\left(\frac{z-1}{z}\right)\left(z+\frac{1}{z}\right)\right]^3=\left(z-\frac{1}{z}\right)^4\left(z^2-\frac{1}{z^2}\right)^3$$

$$=\left(z^4-4z^2+6-\frac{4}{z^2}+\frac{1}{z^4}\right)\left(z^6-3z^2+\frac{3}{z^2}-\frac{1}{z^6}\right)$$

$$=\left(z^{10}-\frac{1}{z^{10}}\right)-4\left(z^8-\frac{1}{z^8}\right)-3\left(z^6-\frac{1}{z^6}\right)+8\left(z^4-\frac{1}{z^4}\right)-14\left(z^2-\frac{1}{z^2}\right)$$

$$=2i\sin10\theta-4(2i\sin8\theta)+3(2i\sin6\theta)+8(2i\sin4\theta)-14(2i\sin2\theta)$$

Since $i^7=-i$

$$\therefore \quad \sin^7\theta\cos^3\theta=-\frac{1}{2^9}[\sin10\theta-4\sin8\theta+3\sin6\theta+8\sin4\theta-14\sin2\theta]$$

2.4 Complex Function

If for each value of the complex variable $z(=x+iy)$ in a given region R, we have one or more values of $\omega(=u+iv)$, then ω is said to be a complex function of z and we write $\omega=u(x,y)+iv(x,y)=f(z)$ where u,v are real functions of x and y.

If to each value of z, there corresponds one and only one value of ω, then ω is said to be a single valued function of z, otherwise a multi valued function. For example, $\omega=\frac{1}{z}$ is a single valued function and $\omega=\sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z=0$ and the latter assumes two values for each value of z except at $z=0$

2.4.1 Exponential Function of a Complex Variable

When x is real, we are already familiar with the exponential function

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^n}{n!}+...\infty$$

Properties

I. Exponential form of $z=re^{i\theta}$. Putting $x=iy$ in (i), we get

$$e^{iy}=1+\frac{iy}{1!}+\frac{(iy)^2}{2!}+\frac{(iy)^3}{3!}+\frac{(iy)^4}{4!}+...\infty=\left(1-\frac{y^2}{2!}+\frac{y^4}{4!}-...\right)+i\left(y-\frac{y^3}{3!}+\frac{y^5}{5!}-...\right)=\cos y+i\sin y$$

Thus $e^z=e^x.e^{iy}=e^x(\cos y+i\sin y)$

Also $x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta}$. Thus, $z=re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i,\left[\because\quad e^{z+2n\pi i}=e^z.e^{2n\pi i}=e^z\right]$

III. e^z is not zero for any value of z

Since $e^z=e^{x+iy}=re^{i\theta}$ or $e^x.e^{iy}=re^{i\theta}\quad\therefore r=e^x>0, y=\theta, \left|e^{iy}\right|=1$

Thus $\left|e^z\right|=\left|e^x\right|.\left|e^{iy}\right|=e^x\neq 0$

IV. $e^{\bar{z}}=\overline{e^z}$. Since $e^{\bar{z}}=e^{x-iy}=e^{-iy}=e^x(\cos y-i\sin y)=\overline{e^x(\cos y+i\sin y)}=\overline{e^z}$

2.4.2 Circular Functions of a Complex Variable

Since $e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos y - i \sin y$

$\therefore$ The circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on}$$

It is, therefore, natural to define the circular functions of the complex variable z by the equations:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}$$

with $\cos ecz, \sec z$ and $\cot z$ as their respective reciprocals.

Properties

I. Circular functions are periodic $\sin z, \cos z$ are periodic functions having real period 2π while $\tan z \cot z$ have period π..$\left[\alpha \sin(z + 2n\pi) = \sin z, \tan(z + n\pi) = \tan z\right]$ etc,

II. Even and odd functions: $\cos z, \sec z$ are even functions while $\sin z, \cos ecz$ are odd functions.

$$\left[\because \quad \cos z = \frac{e^{-iz} + e^{iz}}{2} = \cos z, \text{and} \sin(-z) = \frac{e^{-iz} - e^{iz}}{2i} = \frac{-\left(e^{iz} - e^{-iz}\right)}{2i} = -\sin z\right]$$

III. Zeros of sinz are given by $z = \pm 2n\pi$ and zeros of $\cos z$ are given by $z = \pm \frac{1}{2}(2n+1)\pi, n = 0, 1, 2, \ldots$

IV. All the formulae for real circular functions are valid for complex circular functions

e.g., $\sin^2 z + \cos^2 z = 1, \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$

By definition $\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i \frac{e^{iz} - e^{iz}}{2i} = e^{iz}$ where $z = x + iy$

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \cos\theta + i \sin\theta$, where θ is real or complex. This is called the Euler's theorem.*

Cor. De Moiver's theorem for complex numbers

Whether θ is real or complex, we have

$$\left(\cos\theta + i\sin\theta\right)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example: Prove that

(i) $\left[\sin\left(\alpha+\theta\right) - e^{i\alpha}\sin\theta\right]^n = \sin^n\alpha\, e^{-in\theta}$

(ii) $\sin\left(\alpha - n\theta\right) + e^{-i\alpha}\sin n\theta = e^{-n\theta}\sin\alpha$

Solution: (i) L.H.S. $= \left[\sin\alpha\cos\theta + \cos\alpha\sin\theta - \left(\cos\alpha + i\sin\alpha\right)\sin\theta\right]^n$

$= \left(\sin\alpha\cos\theta - i\sin\alpha\sin\theta\right)^n = \sin^n\alpha\left(\cos\theta - i\sin\theta\right)^n = \sin^n\alpha\left(e^{-i\theta}\right)^n = \sin^n\alpha e^{-in\theta}$

(ii) L.H.S. $= \sin\alpha\cos n\theta - \cos\alpha\sin n\theta + \left(\cos\alpha - i\sin\alpha\right)\sin n\theta$

$= \sin\alpha\cos n\theta - i\sin\alpha\sin n\theta = \sin\alpha\left(\cos n\theta - i\sin n\theta\right) = \sin\alpha.e^{-in\theta}$

2.4.3 Hyperbolic Functions

If x be real or complex,

(i) $\dfrac{e^x - e^{-x}}{2}$ is defined as hyperbolic sine of x and is written as $\sinh x$.

(ii) $\dfrac{e^x + e^{-x}}{2}$ is defined as hyperbolic cosine of x and is written as $\cosh x$

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\sec hx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \cos echx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Properties

I. Periodic functions: $\sinh z$ and $\cosh z$ are periodic functions having imaginary period $2\pi i$.

$\left[\because \sinh\left(z+2\pi i\right)=\sinh z; \cosh\left(z+2\pi i\right)=\cosh z\right]$

II. Even and odd functions: $\cosh z$ is an even function while $\sinh z$ is an odd functions

III. $\sinh 0=0, \cosh 0=1, \tanh 0=0$

IV. Relations between hyperbolic and circular functions.

Since for all values of θ, $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\therefore$ Putting $\theta=ix$, we have $\sin ix=\dfrac{e^{-x}-e^{x}}{2i}=-\dfrac{e^{x}-e^{-x}}{2i}$

$$=i^2\frac{e^{x}-e^{-x}}{2i}=i.\frac{e^{x}-e^{-x}}{2}=i\sinh x$$

and therefore, $\cos ix=\dfrac{e^{-x}+e^{x}}{2}=\cosh x$

thus $\sin ix=i\sinh x$

$\cos ix=\cosh x$

and $\therefore$ $\tan ix=i\tanh x$

Cor. $\cosh ix=\cos x$

$\tanh ix=i\tan x$

2.4.4 Inverse Hyperbolic Functions

If $\sinh u=z$, then u is called the hyperbolic sine inverse of z and is written as $u=\sinh^{-1}z$. Similarly we define $\cosh^{-1}z, \tanh^{-1}z$, etc.

the inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) To show that

(i) $\sinh^{-1}z=\log\left[z+\sqrt{\left(z^2+1\right)}\right]$

(ii) $\cosh^{-1}z=\log\left[z+\sqrt{\left(z^2-1\right)}\right]$, (iii) $\tanh^{-1}z=\dfrac{1}{2}\log\dfrac{1+z}{1-z}$

(i) Let $\sinh^{-1} z = u$, then $z = \sinh u = \frac{1}{2}\left(e^u - e^{-u}\right)$ or $2z = e^u - \frac{1}{e^u}$ or $e^{2u} - 2ze^u - 1 = 0$

This being a quadratic in e^u, we have $e^u = \frac{2z \pm \sqrt{\left(4z^2+4\right)}}{2} = z \pm \sqrt{\left(z^2+1\right)}$

$\therefore$ Taking the positive sign only, we have

$e^u = z + \sqrt{\left(z^2+1\right)} \Rightarrow u = \log\left[z + \sqrt{\left(z^2+1\right)}\right]$

Similarly we can establish (ii)

(iii) Let $\tanh^{-1} z = u$, then $z = \tanh u$; $z = \frac{e^u - e^{-u}}{e^u + e^{-u}}$

Applying componendo and dividendo, we get $\frac{1+z}{1-z} = \frac{e^u}{e^{-u}} = e^{2u}$

$2u = \log\left(\frac{1+z}{1-z}\right)$ whence follows the result

Example: If $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, prove that

(i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

(ii) $\theta = -i\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)$.

Solution: we have $e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ or $\frac{e^{\frac{u}{2}}}{e^{\frac{-u}{2}}} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

By componendo and dividendo, we get

$\frac{e^{\frac{u}{2}} - e^{\frac{-u}{2}}}{e^{\frac{u}{2}} + e^{\frac{-u}{2}}} = \tan\frac{\theta}{2}$ i.e., $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

$$\frac{1}{i}\tan\frac{iu}{2}=\frac{1}{i}\tanh\frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2}=\tanh^{-1}\left(\tan\frac{iu}{2}\right)=\frac{1}{2}\log\frac{1+\tan\frac{iu}{2}}{1-\tan\frac{iu}{2}}$$

$$\theta=\frac{1}{i}\log\tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)=-i\log\tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)$$

Example: Show that $\tanh^{-1}(\cos\theta)=\cosh^{-1}(\cos ec\theta)$

Solution: Let $\tanh^{-1}(\cos\theta)=\phi$ so that $\cos\theta=\tanh\phi$

or $\tanh^2\phi=\cos^2\theta$ or $1-\sec h^2\phi=\cos^2\theta$

or $\sec h^2\phi=1-\cos^2\theta=\sin^2\theta$ or $\sec h\phi=\sin\theta$

or $\cosh\phi=\cos ec\theta$ or $\phi=\cosh^{-1}(\cos ec\theta)$

Example: Find $\tanh x$, if $5\sinh x-\cosh x=5$

Solution: $5\sinh x-\cosh x=5$ (i)

we have $5(\sinh x-1)=\cosh x$

$25(\sinh x-1)^2=\cosh^2 x=1+\sinh^2 x$

$24\sinh^2 x-50\sinh x+24=0$ or $12\sinh^2 x-25\sinh x+12=0$

$(3\sinh x-4)(4\sinh x-3)=0$ whence $\sinh x=\frac{4}{3}$ or $\frac{3}{4}$

$\cosh x=\sqrt{(1+\sinh^2 x)}=\frac{5}{3}$ or $\frac{-5}{4}$ [$\because$ $\cosh x=-\frac{5}{4}$ doesn't satisfy (i)]

Hence $\tanh x=\frac{4}{5}$ or $-\frac{3}{5}$

Example: If $\tan(\theta+i\phi)=e^{i\alpha}$, show that

$$\theta=\left(n+\frac{1}{2}\right)\frac{\pi}{2} \text{ and } \phi=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Solution: Since $\tan(\theta+i\phi)=\cos\alpha+i\sin\alpha$ $\therefore$ $\tan(\theta-i\phi)=\cos\alpha-i\sin\alpha$

$$\tan 2\theta=\tan\left[(\theta+i\phi)+(\theta-i\phi)\right]$$

$$= \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)} = \frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

i.e., $2\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \left(n+\frac{1}{2}\right)\frac{\pi}{2}$

Also $\tan 2i\phi = \tan\left[(\theta+i\phi)-(\theta-i\phi)\right] = \frac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$

or $i\tanh 2\phi = \frac{2i\sin\alpha}{1+(\cos^2\alpha+\sin^2\alpha)} = i\sin\alpha$ or $\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \frac{\sin\alpha}{1}$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1+\sin\alpha}{1-\sin\alpha} = \frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

or $e^{4\phi} = \frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2} = \left(\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right)^2$

or $e^{2\phi} = \frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}} = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$

Example: Separate $\tan^{-1}(x+iy)$ into real and imaginary parts.

Solution: Let $\alpha+i\beta = \tan^{-1}(x+iy)$. Then $\alpha-i\beta = \tan^{-1}(x-iy)$

Adding, $2\alpha = \tan^{-1}(x+iy)+\tan^{-1}(x-iy)^* = \tan^{-1}\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$

$$\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$$

subtracting, $2i\beta = \tan^{-1}(x+iy)-\tan^{-1}(x-iy) = \tan^{-1}\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$

$= \tan^{-1} i\dfrac{2y}{1+x^2+y^2} = i\tanh^{-1}\dfrac{2y}{1+x^2+y^2}$ $\quad\left[\because \quad \tan^{-1} iz = i\tanh^{-1} z\right]$

$\therefore \qquad \beta = \dfrac{1}{2}\tanh^{-1}\dfrac{2y}{1+x^2+y^2}$

Example: Separate $\sin^{-1}\left(\cos\theta + i\sin\theta\right)$ into real and imaginary parts, where θ is a positive acute angle

Solution: Let $\sin^{-1}\left(\cos\theta + i\sin\theta\right) = x + iy$

Then $\cos\theta + i\sin\theta = \sin\left(x+iy\right) = \sin x\cosh y + i\cos x\sinh y$

$\therefore \qquad \cos\theta = \sin x\cosh y \quad$ (i) $\qquad$ and $\sin\theta = \cos x\sinh y \quad$ (ii)

Squaring and adding, we have

$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x\left(1+\sinh^2 y\right) + \cos^2 x\sinh^2 y$

$= \sin^2 x + \sinh^2 y\left(\sin^2 x + \cos^2 x\right)$

or $1-\sin^2 x = \sinh^2 y$, i.e., $\cos^2 x = \sinh^2 y$

Hence from (ii), we have $\sin^2\theta = \cos^4 x$, i.e., $\cos^2 x = \sin\theta$ because θ being a positive acute angle , $\sin\theta$ is positive

As x is to be between $\dfrac{-\pi}{2}$ and $\dfrac{\pi}{2}$, therefore, we have

$\cos x = +\sqrt{\left(\sin\theta\right)}$ $\qquad$ or $x = \cos^{-1}\sqrt{\left(\sin\theta\right)}$

The relation (ii), then, gives $\sinh y = \sqrt{\left(\sin\theta\right)}$

So that $y = \log\left[\sqrt{\left(\sin\theta\right)} + \sqrt{\left(1+\sin\theta\right)}\right]$

2.4.5 Logarithmic Function of a Complex Variable

(1) Definition: If $z(=x+iy)$ and $\omega(=u+iv)$ be so related that $e^{\omega}=z$, then ω is said to be a logarithm of z to the base e and is written as

$$\omega=\log_e z \qquad \text{(i)}$$

also $e^{\omega+2in\pi}=e^{\omega}.e^{2in\pi}=z$ $\qquad \left[\because\ e^{2in\pi}=1\right]$

$$\therefore \qquad \log z=\omega+2in\pi \qquad \text{(ii)}$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi valued function.

The general value of the logarithm of z is written as $\log z$ (beginning with capital L). so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained b taking $n=0$ in $\log z$

Thus from (i) and (ii), $\log(x+iy)=2in\pi+\log(x+iy)$

Obs. 1. If $y=0$, then $Logx=2in\pi+\log x$

This shows that the logarithm of a real quantity is also multi valued. Its principle value is real while all other values are imaginary.

2. We know that the logarithm of a negative quantity has no real value. But we can now evaluate this. e.g.

$$\log_e(-2)=\log_e 2(-1)=\log_e 2+\log_e(-1)=\log_e 2+i\pi=0.6931+i(3.1416)$$

(2) Real and imaginary parts of $L\log(x+iy)$

$$Log(x+iy)=2in\pi+\log(x+iy)$$

$=2in\pi+\log\left[r(\cos\theta+i\sin\theta)\right]$ put $\left[x=r\cos\theta, y=r\sin\theta\right.$ so that $r=\sqrt{(x^2+y^2)}$

and $\left.\theta=\tan^{-1}\left(\frac{y}{x}\right)\right]$

$= 2in\pi + \log\left(re^{i\theta}\right) = 2in\pi + \log r + i\theta == \log\sqrt{\left(x^2+y^2\right)} + i\left[2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right]$

(3) Real and imaginary parts of $\left(\alpha + i\beta\right)^{x+iy}$

$\left(\alpha + i\beta\right)^{x+iy} = e^{(x+iy)Log(\alpha+i\beta)} = e^{(x+iy)\{2in\pi+\log(\alpha+i\beta)\}}$

Put $\left[\alpha = r\cos\theta, \beta = r\sin\theta\right.$ so that $r = \sqrt{\left(\alpha^2+\beta^2\right)}$ and $\left.\theta = \tan^{-1}\frac{\beta}{\alpha}\right.$

$= e^{(x+iy)\left[2in\pi+\log re^{i\theta}\right]} = e^{(x+iy)\left[\log r + i(2n\pi+\theta)\right]}$

$= e^{A+iB} = e^A\left(\cos B + i\sin B\right)$

where $A = x\log r - y\left(2n\pi+\theta\right)$ and $B = y\log r + x\left(2n\pi+\theta\right)$

$\therefore$ the required real part $= e^A\cos B$ and imaginary part $= e^A\sin B$

Example: Find the general value of $Log\left(-i\right)$

Solution: $Log\left(-i\right) = 2in\pi + \log\left[0 + i\left(-1\right)\right]$ put $\left[0 = r\cos\theta, -1 = r\sin\theta \Rightarrow r = 1,\right.$ $\left.\theta = -\frac{\pi}{2}\right]$

$= 2in\pi + \log\left[r\left(\cos\theta + i\sin\theta\right)\right] = 2in\pi + \log\left(re^{i\theta}\right)$

$= 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i\left(\frac{-\pi}{2}\right) = i\left(2n - \frac{1}{2}\right)\pi$

Example: Prove that (i) $i^i = e^{-(4n+1)\frac{\pi}{2}}$ and $Logi^i = -\left(2n+\frac{1}{2}\right)\pi$

(ii) $\left(\sqrt{i}\right)^{\sqrt{i}} = e^{-\alpha}cis\alpha$ where $\alpha = \frac{\pi}{4\sqrt{2}}$

Solution: (i) By definition, we have

$i^i = e^{iLogi} = e^{i(2in\pi+\log i)} = e^{-2n\pi+i\log\left[\exp\left(\frac{i\pi}{2}\right)\right]}$ $\left[\because \quad i = cis\frac{\pi}{2} = \exp\left(\frac{i\pi}{2}\right)\right]$

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$= e^{-2n\pi + i\left(\frac{i\pi}{2}\right)} = e^{-\left(2n+\frac{1}{2}\right)\pi}$

Taking logarithms, we get (ii)

(ii) $\left(\sqrt{i}\right)^{\sqrt{i}} = e^{\sqrt{i}\log\sqrt{i}}$

Now $\sqrt{i}\log\sqrt{i} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\frac{1}{2}}\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\log\left(e^{\frac{i\pi}{2}}\right) = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{2} = \frac{i\pi}{4}\left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\frac{\pi}{4\sqrt{2}}$

Hence $\left(\sqrt{i}\right)^{\sqrt{i}} = e^{-\alpha + i\alpha}$, where $\alpha = \frac{\pi}{4\sqrt{2}}$ $= e^{-\alpha}.e^{i\alpha} = e^{-\alpha}\left(\cos\alpha + i\sin\alpha\right)$

Example: If $\left(a+ib\right)^b = m^{x+iy}$, prove that one of the values of $\frac{y}{x}$ is $2\tan^{-1}\left(\frac{b}{a}\right) \div \log\left(a^2+b^2\right)$

Solution: Taking logarithms, $\left(a+ib\right)^p = m^{x+iy}$ gives $p\log\left(a+ib\right) = \left(x+iy\right)\log m$

$p\left(\frac{1}{2}\log\left(a^2+b^2\right) + i\tan^{-1}\frac{b}{a}\right) = x\log m + iy\log m$

Equating real and imaginary parts from both sides, we get

$\frac{p}{2}\log\left(a^2+b^2\right) = x\log m$ (i) $p\tan^{-1}\frac{b}{a} = y\log m$ (ii)

Division of (ii) by (i) gives $\frac{y}{x} = 2\tan^{-1}\frac{\left(\frac{b}{a}\right)}{\log\left(a^2+b^2\right)}$

Example: If $i^{i^{i^{\ldots\infty}}} = A + iB$ prove that $\tan\frac{\pi A}{2} = \frac{B}{A}$ and $A^2 + B^2 = e^{-B\pi}$

Solution: $i^{i^{i^{\ldots\infty}}} = A + iB$ i.e., $i^{A+iB} = A + iB$

$A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$

$= \exp\left[\left(A+iB\right)\log\left(e^{\frac{i\pi}{2}}\right)\right] = e^{(A+iB)\left(\frac{i\pi}{2}\right)} = e^{\frac{-B\pi}{2}}.e^{\frac{i\pi A}{2}} = e^{\frac{-B\pi}{2}}\left(\cos\frac{\pi A}{2} + i\sin\frac{\pi A}{2}\right)$

Equating real and imaginary parts, we get

$A = e^{\frac{-B\pi}{2}} \cos\frac{\pi A}{2}$ (i) $B = e^{\frac{-B\pi}{2}} \sin\frac{\pi A}{2}$ (ii)

division of (ii) by (i) gives $\frac{B}{A} = \tan\frac{\pi A}{2}$

Squaring and adding (i) and (ii), $A^2 + B^2 = e^{-B\pi}$

Example: Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\frac{b}{a}\right)$. Hence evaluate $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right]$

Solution: Putting $a = r\cos\theta, b = r\sin\theta$ so that $\theta = \tan^{-1}\frac{b}{a}$, we have

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \log\left(e^{i\theta} + e^{-i\theta}\right) = \log e^{2i\theta} = 2i\theta = 2i\tan^{-1}\frac{b}{a}$$

Thus $\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos\left[i(2i\theta)\right] = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2} = \frac{a^2-b^2}{a^2+b^2}$

Example: Separate into real and imaginary parts $\log\sin(x+iy)$

Solution: $\log\sin(x+iy) = \log(\sin x\cos iy + \cos x\sin iy)$

$= \log(\sin x\cosh y + i\cos x\sinh y) = \log r(\cos\theta + i\sin\theta)$

where $r\cos\theta = \sin x\cosh y$ and $r\sin\theta = \cos x\sinh y$

so that $r = \sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y}$

$$= \sqrt{\frac{1-\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2}} = \sqrt{\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right]}$$

and $\theta = \tan^{-1}(\cot x\tanh y)$

thus $\log\sin(x+iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i\tan^{-1}(\cot x\tanh y)$$

Example: Find all the roots of the equation

(i) $\sin z = \cosh 4$ (ii) $\sinh z = i$

Solution: (i) $\sin z = \cosh 4 = \cos 4i = \sin\left(\frac{\pi}{2} - 4i\right)$

$\therefore \quad z = n\pi + (-1)^n \left(\frac{\pi}{2} - 4i\right) \quad \{\because \quad$ If $\sin\theta = \sin\alpha$ then $\theta = n\pi + (-1)^n \alpha$

(ii) $i = \sinh z = \frac{e^z - e^{-z}}{2}$

or $e^{2z} - 2ie^z - 1 = 0$ i.e. $\left(e^z - i\right)^2 = 0$ i.e., $e^z = i$

or $z = Logi = 2in\pi + \log i = 2in\pi + \log e^{\frac{i\pi}{2}} = 2in\pi + \frac{i\pi}{2} = i\left(2n + \frac{1}{2}\right)\pi$

2.5 Summation of Series $C + iS$ Method

This is the most general method and is applied to find the sum of a series of the form

$a_0 \sin\alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + ...$

$a_0 \cos\alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) + ...$

Procedure (i) Put the given series $S =$(or C) according as it is a series of sines for (or cosines)

Then writ e C(or S) = a similar series of cosine (or sines)

If $S = a_0 \sin\alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + ..$

$C = a_0 \cos\alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta)...$

(ii) Multiply the series of series by i and add to the series of cosine, so that

$C + iS = a_0\left[\cos\alpha + i\sin\alpha\right] + a_1\left[\cos(\alpha + \beta) + i\sin(\alpha + \beta)\right] + ...$

$= a_0 e^{ia} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + ...$

(iii) Sum up this last series using any of the following standard series:

(1) Exponential series i.e., $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty = e^x$

(2) Since, cosine, sinh or cosh series

i.e., $x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\infty=\sin x$, $\quad 1-\frac{x^2}{2!}+\frac{x^4}{4!}-...\infty=\cos x$

$x+\frac{x^3}{3!}+\frac{x^5}{5!}+...\infty=\sinh x$, $\quad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+...\infty=\cosh x$

(3) Logarithmic series

$x-\frac{x^2}{2}+\frac{x^3}{3}-...\infty=\log(1+x)$, $\quad -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+...\infty\right)=\log(1-x)$

(4) Gregory's series

$x-\frac{x^3}{3}+\frac{x^5}{5}-...\infty=\tan^{-1}x$, $\quad x+\frac{x^3}{3}+\frac{x^5}{5}+...\infty=\tanh^{-1}x=\frac{1}{2}\log\frac{1+x}{1-x}$

(5) Binomial series

$$1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+...\infty=(1+x)^n$$

$$1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+...\infty=(1+x)^{-n}$$

$$1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+...\infty=(1-x)^{-n}$$

(6) Geometric series

$$a+ar+ar^3+...\text{ to } n \text{ terms}=a\frac{1-r^n}{1-r},\ a+ar+ar^3+...\infty=\frac{a}{1-r},|r|<1$$

(iv) Finally express the sum thus obtained in the form $A+iB$ so that by equating the real and imaginary parts, we get $C=A$ and $S=B$

Series depending on exponential series

Example: Sum the series $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + ...\infty$

Solution: Let $S = \sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + ...\infty$

$$C = \cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + ...\infty$$

$$C + iS = [\cos\alpha + i\sin\alpha] + x[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \frac{x^2}{2!}[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta) + ...\infty]$$

$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!}.e^{i(\alpha+2\beta)} + ...\infty = e^{i\alpha}\left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2e^{2i\beta}}{2!} + ...\infty\right]$$

$$= e^{i\alpha}.e^{xe^{i\beta}} = e^{i\alpha}e^{x(\cos\beta + i\sin\beta)} = e^{x\cos\beta + i(\alpha + x\sin\beta)} = e^{x\cos\beta}e^{i(\alpha + x\sin\beta)}$$

$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i\sin(\alpha + x\sin\beta)]$$

Equating imaginary parts from both sides, we have $S = e^{x\cos\beta}\sin(\alpha + x\sin\beta)$

Series depending on logarithmic series

Example: Sum the series

$$\sin^2\theta - \frac{1}{2}\sin 2\theta\sin^2\theta + \frac{1}{3}\sin 3\theta\sin^3 3\theta - \frac{1}{4}\sin 4\theta\sin^4\theta + ...\infty$$

Solution: Let $S = \sin\theta.\sin\theta - \frac{1}{2}\sin 2\theta.\sin^2\theta + \frac{1}{3}\sin 3\theta.\sin^3\theta - ...\infty$

$$C = \cos\theta.\sin\theta - \frac{1}{2}\cos 2\theta.\sin^2\theta + \frac{1}{3}\cos 3\theta.\sin^3\theta - ...\infty$$

$$C + iS = e^{i\theta}\sin\theta - \frac{e^{2i\theta}\sin^2\theta}{2} + \frac{e^{3i\theta}\sin^3\theta}{3} - ...\infty = \log(1 + e^{i\theta}\sin\theta) = \log[1 + (\cos\theta + i\sin\theta)\sin\theta]$$

$$= \log[1 + \cos\theta\sin\theta + i\sin^2\theta] \qquad [\text{Put } 1 + \cos\theta\sin\theta = r\cos\alpha; \sin^2\theta = r\sin\alpha] \qquad ....(i)$$

$$= \log r(\cos\alpha + i\sin\alpha) = \log re^{i\alpha} = \log r + i\alpha$$

Equating imaginary parts, we have $S = \alpha = \tan^{-1}\left(\frac{\sin^2\theta}{1 + \cos\theta\sin\theta}\right)$

Series depending on binomial series

Example: Find the sum to infinity of the series

$$1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\ldots\left(-\pi<\theta<\pi\right)$$

Solution: Let $C=1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\ldots\infty$

$$S=0-\frac{1}{2}\sin\theta+\frac{1.3}{2.4}\sin 2\theta-\frac{1.3.5}{2.4.6}\sin 3\theta+\ldots\infty$$

$$C+iS=1-\frac{1}{2}e^{i\theta}+\frac{1.3}{2.4}e^{2i\theta}-\frac{1.3.5}{2.4.6}e^{3i\theta}-\ldots$$

$$=1+\left(-\frac{1}{2}\right)e^{i\theta}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}e^{2i\theta}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}e^{3i\theta}+..$$

$$=\left(1+e^{i\theta}\right)^{\frac{-1}{2}}=\left(1+\cos\theta+i\sin\theta\right)^{\frac{-1}{2}}=\left(2\cos^2\frac{\theta}{2}+i.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{\frac{-1}{2}}$$

$$=\left(2\cos\frac{\theta}{2}\right)^{\frac{-1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\frac{-1}{2}}=\left(2\cos\frac{\theta}{2}\right)^{\frac{-1}{2}}\left(\cos\frac{\theta}{4}-i\sin\frac{\theta}{4}\right)$$

Equating real parts, we have $C=\left(2\cos\frac{\theta}{2}\right)^{\frac{-1}{2}}\cos\frac{\theta}{4}$

Approximations and Limits

Example: If $\frac{\sin\theta}{\theta}=\frac{599}{600}$, find an approximate value of θ in radians.

Solutions: Since $\frac{\sin\theta}{\theta}=1-\frac{1}{600}$ which is nearly equal to 1

We know that $\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\ldots$ $\quad\therefore\quad$ $\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{6}+\frac{\theta^4}{5!}$

omitting θ^4 and higher powers, we have

$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{6}=1-\frac{1}{600}$ or $\theta^2=\frac{1}{100}$. Hence $\theta=0.1$ radians.

Example: Solve approximately $\sin\left(\frac{\pi}{6}+\theta\right)=0.51$

Solution: $\sin\left(\frac{\pi}{6}+\theta\right)=\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta=\frac{1}{2}\left(1-\frac{\theta^2}{2!}+\ldots\right)+\frac{\sqrt{3}}{2}\left(\theta-\frac{\theta^3}{3!}+\ldots\right)$

$=\frac{1}{2}+\frac{\sqrt{3}}{2}\theta$, omitting θ^2 and higher powers of θ

Hence the given equation becomes, $\frac{1}{2}+\frac{\sqrt{3}}{2}\theta=0.51$ or $\theta=\frac{1}{50\sqrt{3}}$

$\theta=\frac{1}{50\sqrt{3}}\text{radian}=\frac{\sqrt{3}}{150}\times 57.29$ degree nearly $=39.7^0$

MCQ (Multiple Choice Questions)

Q1. The amplitude of $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

Q2. If $\frac{1-ix}{1+ix}=a+ib$, then a^2+b^2 is

(a)1 (b) -1 (c) 0 (d) none of these

Q3. If $z=1-\cos\theta+i\sin\theta$, then $|z|$ equals

(a) $2\sin\frac{\theta}{2}$ (b) $2\cos\frac{\theta}{2}$ (c) $2\left|\sin\frac{\theta}{2}\right|$ (d) $2\left|\cos\frac{\theta}{2}\right|$

Q4. If $z=\frac{1}{(2+3i)^2}$, then $|z|$ equals

(a) $\frac{1}{13}$ (b) $\frac{1}{15}$ (c) $\frac{1}{12}$ (d) none of these

Q5. If $x_n=\cos\left(\frac{\pi}{2^n}\right)+i\sin\left(\frac{\pi}{2^n}\right)$

Then the value of $x_1.x_2.x_3...,$ up to infinity is

(a) -2 (b) $+2$ (c) -1 (d) $+1$

Q6. If $\left|z-\frac{4}{z}\right|=2$, then the maximum value of $|z|$ is

(a) $\sqrt{5}$ (b) $\sqrt{5}+1$ (c) $\sqrt{5}-1$ (d) none of these

Q7. The value of $1+\sum_{k=0}^{14}\left\{\cos\frac{(2k+1)\pi}{15}+i\sin\frac{(2k+1)\pi}{15}\right\}$ is

(a) 0 (b) -1 (c) 1 (d) i

Q8. The point representing complex number for which $|z+4|^2-|z-4|^2=8$ lie on

(a) a straight line parallel to x-axis

(b) a straight line parallel to y-axis

(c) a circle with centre as origin

(d) a circle with centre other than origin

Q9. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x+iy$, then

(a) $x=3, y=1$ (b) $x=1, y=3$

(c) $x=0, y=3$ (d) $x=0, y=0$

Q10. If the number $\frac{z-1}{z+1}$ is purely imaginary, then

(a) $|z|=1$ (b) $|z|>1$ (c) $|z|<1$ (d) $|z|>2$

NAT (Numerical Answer Type)

Q11. The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n$ is real is…

Q12. The value of $i^n+i^{n+1}+i^{n+2}+i^{n+3}$ is

Q13. The value of $\frac{(\cos 3\theta+i\sin 3\theta)^4(\cos 4\theta-i\sin 4\theta)^5}{(\cos 4\theta+i\sin 4\theta)^3(\cos 5\theta+i\sin 5\theta)^{-4}}$ is

Q14. For $n=6k, k\in z$,

$\left(\frac{1-i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n$ has the value…

Q15. If $z=\frac{\sqrt{3}+i}{2}$, then z^{69} equals_____.

MSQ (Multiple Select Questions)

Q16. If $z = 4 + 2i$, then

(a) The magnitude of z is $2\sqrt{5}$ units

(b)The magnitude of z is $3\sqrt{5}$ units

(c) The argument or amplitude of z is $\tan^{-1}\frac{1}{3}$

(d) The argument or amplitude of z is $\tan^{-1}\frac{1}{2}$

Q17. Consider a complex number $z = 1 - \sqrt{3}\, i$

(a) The magnitude of z is 2 units

(b) The magnitude of z is 4 units

(c) The complex number z can be represented by a point $\left(1, -\sqrt{3}\right)$ in the complex or Argand plane

(d) The complex number z can be represented by a point $\left(-\sqrt{3}, -1\right)$ in the complex plane.

Q18. If $z_1 = 2 + 3i$ and $z_2 = 1 + 2i$ then

(a) $z_1 + z_2$ is equal to $4 + 6i$

(b) $z_1 - z_2$ is equal to $1 + i$

(c) $z_1.z_2$ is equal to $-4 + 7i$

(d) $\frac{z_1}{z_2}$ is equal to $\frac{8}{5} - \frac{1}{5}i$

Q19. Pick out the correct statement (S)

(a) Addition of two complex number is commutative

(b) For any three complex numbers z_1, z_2 and z_3 we have $\left(z_1 + z_2\right) + z_3 = z_1 + \left(z_2 + z_3\right)$

(c) For any two complex numbers z_1 and z_2 we have $z_1 \cdot z_2 = z_2 \cdot z_1$

(d) Multiplication of two complex number are not commutative

Q20. Pick out the correct statement(s)

(a) Let z and $\bar{z}$ denote a complex number and its conjugate respectively then, $z = \bar{z}$ implies z is purely imaginary

(b) Let z and $\bar{z}$ denote a complex number and its conjugate respectively then, $z + \bar{z} = 0$ implies z is purely imaginary

(c) In polar form the complex number $z = 1 + i$ can be written as

$$\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(d) In polar form the complex number $1 + i$ can be written as

$$\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

fiziks

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: $\frac{1+i\sqrt{3}}{\sqrt{3}+i}=\frac{\left(1+i\sqrt{3}\right)\left(\sqrt{3}-i\right)}{\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)}=\frac{2\sqrt{3}+2i}{4}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$

Since both the real and complex parts are greater than zero, hence the argument is the acute angle given by $\tan^{-1}\left|\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$

Ans. 2: (a)

Solution: $a+ib=\frac{1-ix}{1+ix}\Rightarrow a-ib=\frac{1+ix}{1-ix}$

$\therefore\quad \left(a+ib\right)\left(a-ib\right)=\frac{1-ix}{1+ix}.\frac{1+ix}{1-ix}\Rightarrow a^2+b^2=\frac{1+x^2}{1+x^2}=1$

Ans. 3: (c)

Solution: $\left|z\right|=\sqrt{\left(1-\cos\theta\right)^2+\sin^2\theta}=\sqrt{2-2\cos\theta}=\sqrt{4\sin^2\frac{\theta}{2}}=2\left|\sin\frac{\theta}{2}\right|$

Ans. 4: (a)

Solution: $\left|z\right|=\frac{1}{\left|2+3i\right|^2}=\frac{1}{\left(\sqrt{2^2+3^2}\right)^2}\qquad \left|z\right|=\frac{1}{13}$

Ans. 5: (c)

Solution: $x_1.x_2.x_3....$ to infinity $=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3}+i\sin\frac{\pi}{2^3}\right)...\infty$

$=\cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+...\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+...\right)=\cos\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right)+i\sin\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right)$

$=\cos\pi+2i\sin\pi=-1$

Ans. 6: (b)

Solution: $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \left|\frac{4}{z}\right|$

Using triangle in equinity

$\Rightarrow \quad |z| \le 2 + \frac{4}{|z|} \qquad$ since $\left|z - \frac{4}{z}\right| = 2$

$\Rightarrow \quad |z|^2 - 2|z| - 4 \le 0 \Rightarrow \left(|z| - 1 + \sqrt{5}\right)\left(|z| - 1 - \sqrt{5}\right) \le 0 \Rightarrow \left(1 - \sqrt{5}\right) \le |z| \le 1 + \sqrt{5}$

Thus the maximum value of $|z|$ is $1 + \sqrt{5}$

Ans. 7: (c)

Solution: $1 + \sum_{k=0}^{14} \left\{ \cos \frac{(2k+1)\pi}{15} + i \sin \frac{(2k+1)\pi}{15} \right\}$

$= 1 + \sum_{k=0}^{14} e^{i\frac{(2k+1)\pi}{15}} = 1 + \sum_{k=0}^{14} \alpha^{2k+1}, \qquad$ where $\alpha = e^{i\frac{\pi}{15}}$

$= 1 + \left(\alpha + \alpha^3 + \alpha^5 + \ldots \alpha^{29}\right) = 1 + \alpha\left(\frac{1 - \left(\alpha^2\right)^{15}}{1 - \alpha^2}\right)$

$= 1 + \alpha\left(\frac{1 - \alpha^{30}}{1 - \alpha^2}\right) = 1 + \alpha\left(\frac{1 - 1}{1 - \alpha^2}\right) = 1 \qquad \left[\text{since } \alpha^{30} = e^{i2\pi} = 1\right]$

Ans. 8: (b)

Solution: Let $z = x + iy$, then

$|z + 4|^2 - |z - 4|^2 = 8 \Rightarrow 4\,\mathrm{Re}(4z) = 8$

where $\mathrm{Re}(4z)$ is the real part of the complex number $4z$

$\Rightarrow \quad \mathrm{Re}(z) = \frac{1}{2} \Rightarrow x = \frac{1}{2}$, which is a straight line parallel to the y-axis.

Ans. 9: (d)

Solution: we have

$$\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = \begin{vmatrix} 6i & 0 & 1 \\ 4 & 0 & -1 \\ 20 & 0 & i \end{vmatrix} \qquad (\text{Applying } C_2 \to C_2 + 3iC_2)$$

$$= 0 = 0 + 0i$$

$$\therefore x = 0, y = 0$$

Ans. 10: (a)

Solution: we have: $\frac{z-1}{z+1}$ is purely imaginary

$\Rightarrow$ argument of $\frac{z-1}{z+1}$ is $\pm\frac{\pi}{2} \Rightarrow \arg\left(\frac{z-1}{z+1}\right) = \pm\frac{\pi}{2}$

$\Rightarrow$ z lies on a circle having $(1,0)$ and $(-1,0)$ as the end point of a diameter.

$\Rightarrow$ z lies on a circle with centre at the origin and radius are unit

$\Rightarrow$ z lies on $|z| = 1 \Rightarrow |z| = 1$

NAT (Numerical Answer Type)

Ans. 11: 2

Solution: we have

$\left(\frac{1+i}{1-i}\right)^n = \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^n = \left(\frac{(1+i)^2}{1-i^2}\right)^n = i^n$. Clearly, it is real for $n = 2$

Ans. 12: 0

Solution: $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n\left(1 + i + i^2 + i^3\right)$

$= i^n\left[1 + i - 1 - i\right] = 0$

Ans. 13: 1

Solution: we have

$$\left(\cos 3\theta + i\sin 3\theta\right)^4 = \cos 12\theta + i\sin 12\theta = \left(\cos\theta + i\sin\theta\right)^{12}$$

$$(\cos 4\theta - i\sin 4\theta)^5 = \cos 20\theta - i\sin 20\theta = (\cos\theta + i\sin\theta)^{-20}$$

$$(\cos 4\theta + i\sin 4\theta)^3 = \cos 12\theta + i\sin 12\theta = (\cos\theta + i\sin\theta)^{12}$$

$$(\cos 5\theta + i\sin 5\theta)^{-4} = \cos 20\theta - i\sin 20\theta = (\cos\theta + i\sin\theta)^{-20}$$

Hence the value of given expression is 1

Ans. 14: 2

Solution: $\frac{1-i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{3}$ are cube roots of unity. If we denote $\frac{1-i\sqrt{3}}{2}$ by ω then $\frac{-1-i\sqrt{3}}{3} = \omega^2$

Hence $\left(\frac{1-i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n = \omega^n + (\omega^2)^n = \omega^{6k} + \omega^{12k} = (\omega^3)^{2k} + (\omega^3)^{4k} = 1+1 = 2$

Ans. 15: $-i$

Solution: $z^{69} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{69} = \left(\cos\frac{69\pi}{6} + i\sin\frac{69\pi}{6}\right) = \cos\frac{23\pi}{2} + i\sin\frac{69\pi}{6}$

$= i\sin\frac{23\pi}{2} = -i\sin\frac{\pi}{2} = -i$

MSQ (Multiple Select Questions)

Ans. 16: (a), (d)

Solution: The magnitude of a complex number

$z = x + iy$ is given by $|z| = \sqrt{x^2 + y^2}$

hence $z = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$ units

The argument is given by

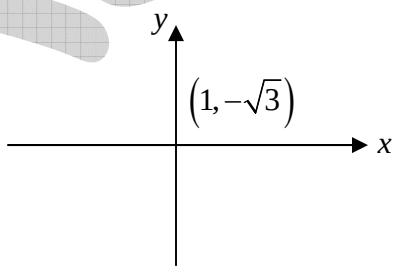
$\tan^{-1}\left|\frac{y}{x}\right|$, since both x & $y > 0$

Hence argument or amplitude is $z = \tan^{-1}\left|\frac{2}{4}\right| = \tan^{-1}\frac{1}{2}$

Ans. 17: (a) and (c)

Solution: The magnitude is $|z| = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2} = 2$ units.

A complex number in the complex plane is represent by a point. The real part is plotted on the x-axis and imaginary part on the y-axis. hence $z = 1 - \sqrt{3}i$ can be represented by a point $\left(1, -\sqrt{3}\right)$ in the complex plane.



Ans. 18: (b), (c) and (d)

Solution: $z_1 + z_2 = (2 + 3i) + (1 + 2i) = 3 + 5i$

$z_1 - z_2 = (2 + 3i) - (1 + 2i) = 1 + i$

$z_1.z_2 = (2 + 3i).(1 + 2i) = 2 + 4i + 3i - 6$ (since $i^2 = -1$)

$= -4 + 7i$

$\frac{z_1}{z_2} = (2 + 3i).\frac{1}{(1 + 2i)}$, but $\frac{1}{1 + 2i} = \frac{1 - 2i}{5} = \frac{1}{5} - \frac{2}{5}i$

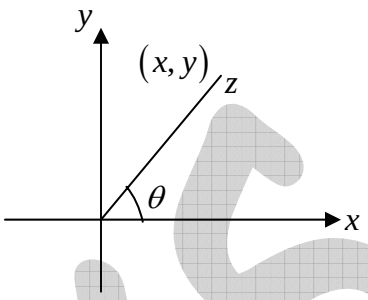
hence $\frac{z_1}{z_2} = (2 + 3i)\left(\frac{1}{5} - \frac{2}{5}i\right) = \left(\frac{2}{5} + \frac{6}{5}\right) + i\left(-\frac{4}{5} + \frac{3}{5}\right) = \frac{8}{5} - \frac{1}{5}i$

Ans. 19: (a), (b) and (c)

Solution: The first three options are true. Option (d) is false as multiplication of two complex number is commutative

Ans. 20: (b) and (c)

Solution: Let $z = x + iy$, then $\bar{z} = x - iy$



$z = \bar{z} \Rightarrow x + iy = x - iy \Rightarrow 2iy = 0$ or $y = 0$

Thus z is purely real

$z + \bar{z} = 0 \Rightarrow x + iy + x - iy = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

Thus z is purely imaginary

Any complex number in the complex plane can be written as

$z = r(\cos\theta + i\sin\theta)$

where r is the modulus of the complex number and θ is the angle as shown in the figure

$r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\tan\theta = \dfrac{\text{Imaginary part}}{\text{real part}} = \dfrac{1}{1} = 1$

$\therefore \quad \theta = \dfrac{\pi}{4}$

Hence the complex number $z = 1 + i$ can be written as

$z = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

3. Fourier Series

Any piecewise smooth function defined on a finite interval has a Fourier series expansion. A Fourier series is defined as an expansion of a periodic function or representation of a function in a series of sines and cosines, such as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad \ldots\ldots (1)$$

The coefficients a_0, a_n a and b_n are the Fourier coefficients of $f(x)$ defined as

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$$

This result, of course, is subject to the requirement that the integrals exist. They do if $f(x)$ is piecewise continuous (or square integrable). Notice that a_0 is singled out for special treatment by the inclusion of the factor $1/2$. This is done so that equation (2) will apply to all $a_n, n=0$ as well as $n>0$.

The conditions imposed on $f(x)$ to make Eq. (1) valid, and the series convergent, are that $f(x)$ has only a finite number of finite discontinuities and only a finite number of extreme values, maxima, and minima in the interval $[0,2\pi]$. Functions satisfying these conditions are called **piecewise regular**. The conditions are known as the **Dirichlet conditions**.

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$

The $\{a_n, b_n\}$ are the Fourier coefficients of $f(x)$.

Note that the cosine functions (and the function 1) are even, while the sine functions are odd.

If $f(x)$ is even $f(-x) = +f(x)$ for all x, then $b_n = 0$ for all n, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.

If $f(x)$ is odd $f(-x) = -f(x)$ for all x, then $a_n = 0$ for all n, leaving a Fourier sine series only for $f(x)$.

3.1 Half-Range Fourier Series

A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.

An odd extension leads to a **Fourier sine series**: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

where $b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$

An even extension leads to a **Fourier cosine series**:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and there is automatic continuity of the Fourier cosine series at $x = 0$ and at $x = \pm L$.

The **Fourier series** of a periodic function $f(t)$ of period T can be written as a sum of sines and cosines as

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)}$$

Where, a_0 represents the average value of the function, and is calculated by:

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

The a_n coefficients can be calculated by the formula: $\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)dt}$

The b_n coefficients can be calculated by the formula: $\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt}$

Example: Find the Fourier series representing

$$f(x) = x, \qquad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = -4\pi$

Solution: Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + ...$

Hence $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \quad dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx \quad dx$$

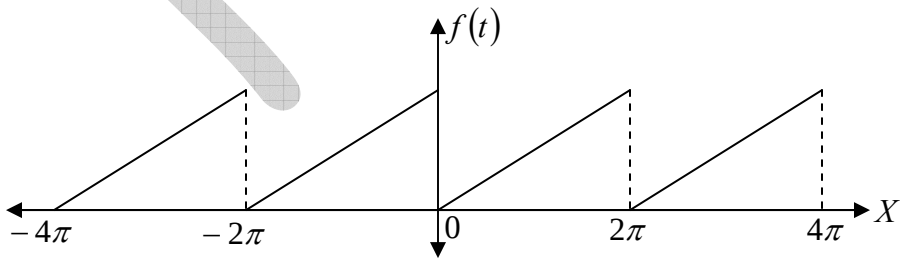
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \quad dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nxdx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[-\frac{2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\right]$$



Example: Given that $f(x)=x+x^2$ for $-\pi<x<\pi$, find the Fourier expression of $f(x)$.

Deduce that $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....$

Solution: Let $x+x^2=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+....+b_1\sin x+b_2\sin 2x+...$ ---(1)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx$$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\ \ dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\ \ dx$$

$$=\frac{1}{\pi}\left[\left(x+x^2\right)\frac{\sin nx}{n}-(2x+1)\left(-\frac{\cos nx}{n^2}+(2)\left(-\frac{\sin nx}{n^3}\right)\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2}-(-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right]=\frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right]=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\left(x+x^2\right)\left(-\frac{\cos nx}{n}\right)-(2x+1)\left(-\frac{\sin nx}{n^2}\right)+2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]=-\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$x+x^2=\frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+....\right]$$

$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+....\right] \quad ---(2)$$

Put $x=\pi$ in (2), $\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$.....(3)

Put $x=-\pi$, $-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$.....(4)

Adding (3) and (4) $2\pi^2=\frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$

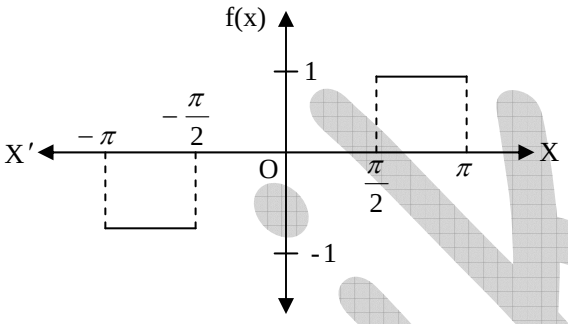
$$\frac{4\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....=\sum_{n=1}^{\infty}\frac{1}{n^2}$$.

3.2 Functions defined in two or more sub-ranges

Example: Find the Fourier series of the function $f(x)=\begin{cases}-1 & \text{for } -\pi<x<-\frac{\pi}{2}\\ 0 & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ +1 & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$

Solution:



Let $f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+.....+b_1\sin x+b_2\sin 2x+....$ ---(1)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}}(-1)dx+\frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}0dx+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(1)dx \Rightarrow a_0=\frac{1}{\pi}\left[-\left(\frac{-\pi}{2}+\pi\right)+\left(\pi-\frac{\pi}{2}\right)\right]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}}-\cos nxdx+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos nxdx \Rightarrow a_n=\frac{-1}{\pi}\frac{\sin nx}{n}\Big|_{-\pi}^{\frac{-\pi}{2}}+\frac{1}{\pi}\frac{\sin nx}{n}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$a_n=\frac{-1}{\pi}\left(\frac{-1}{n}\right)+\frac{1}{\pi}\frac{(-1)}{n}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}}-\sin nxdx+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin nxdx \Rightarrow b_n=\frac{-1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{\frac{-\pi}{2}}+\frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$b_n=\frac{1}{n\pi}\left[-(-1)^n\right]-\frac{1}{n\pi}(-1)^n=\frac{-2(-1)^n}{n\pi}$$

Example: Find the Fourier series for $f(x)$, if $f(x)=\begin{cases}-\pi & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.....=\frac{\pi^2}{8}$

Solution: Let $f(x)$ $f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+....+a_n\cos nx+....$

$+b_1\sin x+b_2\sin 2x+....+b_n\sin nx+....$ ---(1)

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$

Then $a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)dx+\int_0^{\pi}x\,dx\right]=\frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0}+\left(x^2/2\right)_0^{\pi}\right]=\frac{1}{\pi}\left(-\pi^2+\pi^2/2\right)=-\frac{\pi}{2};$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\;\;dx+\int_0^{\pi}x\cos\;\;nx\;\;dx\right]=\frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right)_0^{\pi}\right]$

$=\frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}\right]=\frac{1}{\pi n^2}(\cos n\pi-1);$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\;\;dx+\int_0^{\pi}x\sin nx\;\;dx\right]=\frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$

$=\frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi)-\frac{\pi}{n}\cos n\pi\right]=\frac{1}{\pi}(1-2\cos n\pi)$

$f(x)=-\frac{\pi}{4}-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+...\right)+3\sin x-\frac{\sin 2x}{2}+\frac{3\sin 3x}{3}-\frac{\sin 4x}{4}$ ---(2)

Putting $x = 0$ in (2), we get $f(0)=-\frac{\pi}{4}-\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+....\infty\right)$

Now $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0 \therefore f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

From (3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$ or $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$

Example: Find the Fourier series expansion of the periodic function of period 2π

$$f(x) = x^2, \qquad -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$

Solution: $f(x) = x^2, \qquad -\pi \le x \le \pi$

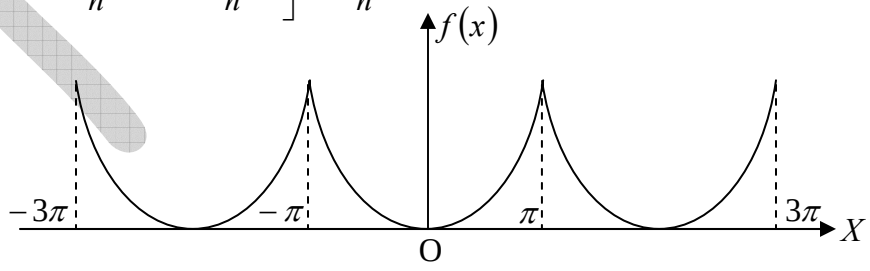
This is an even function $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2inn\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is $f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+a_3\cos 3x+....+a_n\cos nx+....$

$$x^2=\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+....\right]$$

on putting x = 0, we have $0=\frac{\pi^2}{3}-4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}....\right]$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}....=\frac{\pi^2}{12}$$

Example: Find the Fourier sine series for the function

$$f(x)=e^{ax} \quad \text{for} \quad -\pi\le x\le\pi$$

where a is constant.

Solution: $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$

$$b_n=\frac{2}{\pi}\int e^{ax}\sin nx\,dx$$

$$=\frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx-n\cos nx)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\pi-n\cos n\pi)+\frac{n}{a^2+n^2}\right]$$

$$=\frac{2}{\pi}\left(\frac{n}{a^2+n^2}\right)\left[-(-1)^n e^{a\pi}+1\right]=\frac{2n}{(a^2+n^2)\pi}\left[1-(-1)^n e^{a\pi}\right]$$

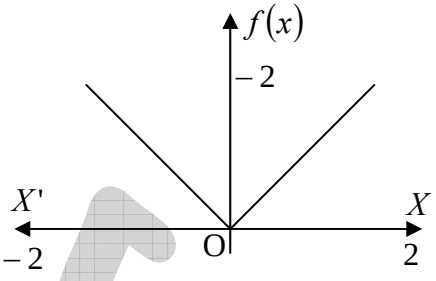
$$b_1=\frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \qquad b_2=\frac{2.2.(1-e^{a\pi})}{(a^2+2^2)\pi}$$

$$e^{ax}=\frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x+\frac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x+....\right]$$

Example: A periodic function of period 4 is defined as

$$f(x)=|x|, \quad -2<x<2$$

Find its Fourier series expansion



Solution: $f(x)=|x| \qquad -2<x<2$

$$f(x)=-x \qquad -2<x<0$$

$$=x \qquad 0<x<2$$

$$a_0=\frac{1}{c}\int_{-c}^{c} f(x)\,dx=\frac{1}{2}\int_{-2}^{0}(-x)\,dx+\frac{1}{2}\int_{0}^{2}x\,dx$$

$$=\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2+\frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^0=\frac{1}{4}(4-0)+\frac{1}{4}(0+4)=2$$

$$a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx=\frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx+\frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$=\frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^0$$

$$=\frac{1}{2}\left[0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right]+\frac{1}{2}\left[0-\frac{4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$=\frac{1}{2}\frac{4}{n^2\pi^2}\left[(-1)^n-1-1+(-1)^n\right]=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$=-\frac{8}{n^2\pi^2} \qquad \text{if } n \text{ is odd}$$

$$=0 \qquad \text{if } n \text{ is even}$$

$b_n = 0$ since $f(x)$ is even function

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{c} + a_2 \cos\frac{2\pi x}{c} + ... + b_1 \sin\frac{\pi x}{c} + b_2 \sin\frac{2\pi x}{c} + ...$$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2}\right]$$

Example: Find the Fourier half-range cosine series of the function

$$f(t) = 2t \qquad 0 < t < 1$$
$$= 2(2-t) \qquad 1 < t < 2$$

Solution: $f(t) = 2t \qquad 0 < t < 1$

$$= 2(2-t) \qquad 1 < t < 2$$

Let $$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{c} + a_2 \cos\frac{2\pi t}{c} + a_3 \cos\frac{3\pi t}{c} + ...$$

$$+ b_1 \sin\frac{\pi t}{c} + b_2 \frac{2\pi t}{c} + b_3 \sin\frac{3\pi t}{c} + ... \qquad \text{----(1)}$$

Hence $c = 2$, because it is half range series.

Hence $a_0 = \frac{2}{c}\int_0^c f(t)dt = \frac{2}{2}\int_0^1 2t\,dt + \frac{1}{2}\int_1^2 2(2-t)dt$

$$= \left[t^2\right]_0^1 + \left[2\left(2\ \ t - \frac{t^2}{2}\right)\right]_1^2 = 1 + (8 - 4 - 4 + 1) = 2$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{2}dt = \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}dt + \frac{2}{2}\int_1^2 2(2-t)\cos\frac{n\pi t}{2}dt$$

$$= \left[2t\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_0^1$$

$$+\left[(4-2t)\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right)-(-2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_1^2$$

$$=\left[\frac{4}{n\pi}\sin\frac{n\pi}{2}+\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{8}{n^2\pi^2}\right]+\left[0-\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{4}{n\pi}\sin\frac{n\pi}{2}+\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$=\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{8}{n^2\pi^2}-\frac{4}{n\pi}\sin\frac{n\pi}{2}=\frac{8}{n^2\pi^2}\left[\cos\frac{n\pi}{2}-1-\sin\frac{n\pi}{2}\sin\frac{n\pi}{2}\right]$$

If $n=1$, $\quad a_1=\frac{8}{\pi^2}\left[0-1-\frac{\pi}{2}\right]=-\frac{8}{\pi^2}-\frac{4}{\pi}$

If $n=2$, $\quad a_2=\frac{8}{4\pi^2}[-1-1]=-\frac{16}{4\pi^2}=-\frac{4}{\pi^2}$

If $n=3$, $\quad a_3=\frac{8}{9\pi^2}\left[0-1-\frac{3\pi}{2}\right]=-\frac{8}{9\pi^2}-\frac{4}{3\pi}$

Putting the values of $a_0,a_1,a_2,a_3\ldots$ in (1) we get

$$f(t)=1-\left(\frac{8}{\pi^2}+\frac{4}{\pi}\right)\cos\frac{\pi t}{2}-\frac{4}{\pi^2}\cos\frac{2\pi t}{2}+\left(-\frac{8}{9\pi^2}+\frac{4}{3\pi}\right)\cos\frac{3\pi t}{2}+\ldots.$$

Example: Find the Fourier series expansion of the periodic function of period 1.

$$f(x)=\frac{1}{2}+x,\quad -\frac{1}{2}<x\le 0$$

$$=\frac{1}{2}-x,\qquad 0<x<\frac{1}{2}$$

Solution: Let $\quad f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{c}+a_2\cos\frac{2\pi x}{c}+\ldots.$

$$+b_1\sin\frac{\pi x}{c}+b_2\sin\frac{2\pi x}{c}+\ldots. \qquad \text{---(1)}$$

Here $\qquad 2c=1 \qquad$ or $\qquad c=\frac{1}{2}$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)dx$$

$$= 2\left[\frac{x}{2}+\frac{x^2}{2}\right]_{-1/2}^{0} + 2\left[\frac{x}{2}-\frac{x^2}{2}\right]_{0}^{1/2} = 2\left[\frac{1}{4}-\frac{1}{8}\right]+2\left[\frac{1}{4}-\frac{1}{8}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos\frac{n\pi x}{1/2}dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos\frac{n\pi x}{1/2}dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\frac{\sin 2n\pi x}{2n\pi} - (1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+2\left[\left(\frac{1}{2}-x\right)\frac{\sin 2n\pi x}{2n\pi} - (-1)\left(\frac{-\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[0+\frac{1}{4n^2\pi^2}-\frac{(-1)^n}{4n^2\pi^2}\right]+2\left[0-\frac{(-1)^n}{4n^2\pi^2}+\frac{1}{4n^2\pi^2}\right] = \frac{1}{\pi^2}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{n^2\pi^2} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin\frac{n\pi x}{c}dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin\frac{n\pi x}{1/2}dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi x\,dx + 2\int\left(\frac{1}{2}-x\right)\sin 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right)-(-1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+2\left[\left(\frac{1}{2}-x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right)-(-1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

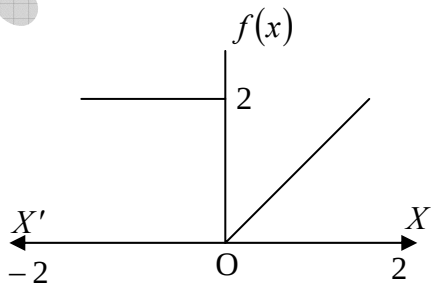
$$= 2\left[-\frac{1}{4n\pi}\right]+2\left[\frac{1}{4n\pi}\right]=0$$

Substituting the values of $a_0, a_1, a_2, a_3 \ldots b_1, b_2, b_3 \ldots$ in (1) we have

$$f(x)=\frac{1}{4}+\frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2}+\frac{\cos 6\pi x}{3^2}+\frac{\cos 10\pi x}{5^2}+\ldots.\right]$$

Example: Find the Fourier series corresponding to the function $f(x)$ defined in (- 2, 2) as follows

$$f(x)=2 \quad \text{in} \quad -2\le x\le 0$$
$$=x \quad \text{in} \quad 0<x<2$$



Solution: Here the interval is (- 2, 2) and c = 2

$$a_0=\frac{1}{c}\int_{-c}^{c} f(x)dx=\frac{1}{2}\left[\int_{-2}^{0}2.dx+\int_{0}^{2}x.dx\right]$$

$$=\frac{1}{2}\left[[2x]_{-2}^{0}+\left(\frac{x^2}{2}\right)_{0}^{2}\right]=\frac{1}{2}[4+2]=3$$

$$a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}2\cos\frac{n\pi x}{2}dx+\int_{0}^{2}x\cos\frac{n\pi x}{2}dx\right]$$

$$=\frac{1}{2}\left[\frac{4}{n\pi}\left(\sin\frac{n\pi x}{2}\right)_{-2}^{0}+\left(x\frac{2}{n\pi}\sin\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= -\frac{4}{n^2\pi^2} \qquad \text{when } n \text{ is odd}$$

$$= 0 \qquad \text{when } n \text{ is even}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{c}dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + (1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_{0}^{2}$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right] = \frac{1}{2}\left[-\frac{4}{n\pi}\right] = -\frac{2}{n\pi}$$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{2} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + \ldots$$

$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + b_3\sin\frac{3\pi x}{c} + \ldots$$

$$= \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots.\right\}$$

$$-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\right\}$$

Example: Expand $f(x) = e^x$ in a cosine series over (0, 1).

Solution: $f(x) = e^x$ and $c = 1$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 e^x dx = 2(e-1)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}\left(n\pi\sin n\pi x+\cos n\pi x\right)\right]_0^1 = 2\left[\frac{e^x}{n^2\pi^2+1}\left(n\pi\sin n\pi+\cos n\pi\right)-\frac{1}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}\left[(-1)^n e-1\right]$$

$$f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + ...$$

$$e^x = e-1+2\left[\frac{-e-1}{\pi^2+1}\cos\pi x+\frac{e-1}{4\pi^2+1}\cos 2\pi x+\frac{-e-1}{9\pi^2+1}\cos 3\pi x+.....\right]$$

Example: By using the sine series for $f(x) = 1$ in $0 < x < \pi$ show that

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+....$$

Solution: sine series is $f(x) = \Sigma b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi (1)\sin nx\;\; dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^\pi = \frac{-2}{n\pi}\left[\cos n\pi-1\right] = \frac{-2}{n\pi}\left[(-1)^n-1\right]$$

$$= \frac{4}{n\pi} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

Then, the sine series is

$$1 = \frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin 3x+\frac{4}{5\pi}\sin 5x+\frac{4}{7\pi}\sin 7x+....$$

$$\int_0^c\left[f(x)\right]^2 dx = \frac{c}{2}\left[b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+....\right]$$

$$\int_0^\pi (1)^2\,dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2+\left(\frac{4}{3\pi}\right)^2+\left(\frac{4}{5\pi}\right)^2+\left(\frac{4}{7\pi}\right)^2+....\right]$$

$$[x]_0^{\pi} = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....\right]$$

$$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....$$

Example: Expand

$$f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$$ in a Fourier series.

Solution: $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx - \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

Therefore the Fourier series for $f(x)$ is

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)$$

3.3 Complex Notation for Fourier series

The Fourier series $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

can be written in complex form as

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx} \qquad \text{where } c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$$

and

$$c_0=\frac{1}{2}a_0 \qquad c_n=\frac{1}{2}(a_n-ib_n) \qquad c_{-n}=\frac{1}{2}(a_n+ib_n) \qquad n=1,2,3,\ldots$$

$$a_0=2c_0 \qquad a_n=(c_n+c_{-n}) \qquad b_n=i(c_n-c_{-n}) \qquad n=1,2,3,\ldots$$

Properties of Fourier Series

Convergence

It might be noted, first that our Fourier series should not be expected to be uniformly convergent if it represents a discontinuous function. A uniformly convergent series of continuous function $(\sin nx,\cos nx)$ always yields a continuous function. If, however,

(a) $f(x)$ is continuous, $-\pi\le x\le\pi$;

(b) $f(-\pi)=f(\pi)$;

(c) $f'(x)$ is sectionally continuous,

The Fourier series for $f(x)$ will converge uniformly. These restrictions do not demand that $f(x)$ be periodic, but they will satisfied by continuous, differentiable, periodic function (period of 2π).

Integration

Term-by-term integration of the series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \qquad \text{..... (a)}$$

yields $$\int_{x_0}^{x} f(x)dx=\frac{a_0 x}{2}\Big|_{x_0}^{x}+\sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx\Big|_{x_0}^{x}-\sum_{n=1}^{\infty}\frac{b_n}{n}\cos nx\Big|_{x_0}^{x}.$$

Clearly, the effect of integration is to place an additional power of n in the denominator of each coefficient. This results in more rapid convergence than before. Consequently, a convergent Fourier series may always be integrated term by term, the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series is not itself convergent! The function $f(x)$ need only be integrable.

Strictly speaking, Eq. (a) may be a Fourier series; that is , if $a_0\neq 0$, there will be a term $\frac{1}{2}a_0x$. However, $\int_{x_0}^{x} f(x)-\frac{1}{2}a_0x$ will still be a Fourier series.

Differentiation

The situation regarding differentiation is quite different from that of integration. Consider the series for

$$f(x)=x, \qquad -\pi<x<\pi.$$

We readily find that the Fourier series is

$$x=2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}, \qquad -\pi<x<\pi.$$

Differentiating term by term, we obtain $1=2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx,$ which is not convergent.

For a triangular wave which the convergence is more rapid (and uniform),

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,odd}^{\infty}\frac{\cos nx}{n^2}.$$

Differentiating term by term

$$f'(x)=\frac{4}{\pi}\sum_{n=1,odd}^{\infty}\frac{\sin nx}{n}.$$

which is the Fourier expansion of a square wave

$$f'(x)=\begin{cases}1, & 0<x<\pi,\\ -1, & -\pi<x<0.\end{cases}$$

As the inverse of integration, the operation of differentiation has placed an additional factor n in the numerator of each term. This reduces the rate of convergence and may, as in the first case mentioned, render the differentiated series divergent.

In general, term-by-term differentiation is permissible under the same conditions listed for uniform convergence.

MCQ (Multiple Choice Questions)

Q1. A Fourier series is defined as an expansion of a function in a series of sines and cosines,

such as $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Consider a function $f(x)=x+x^2$ for $-\pi<x<\pi$. Which of the following is correct statement?

(a) $\frac{\pi^2}{6}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.....$

(b) $\frac{\pi^2}{6}=-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-.....$

(c) $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....$

(d) $\frac{\pi^2}{6}=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right)$.

Q2. Consider the function $f(x)=\begin{cases}-k & -\pi<x\le 0,\\ +k & 0<x<\pi\end{cases}$ and $f(x+2\pi)=f(x)$

The Fourier series which represents the above function has

(a) Sine terms only

(b) Cosine terms only

(c) Sine terms and constant

(d) Cosine terms and constant.

Q3. The function $f(x)=x^2;\ -\pi<x<\pi$ has a Fourier series representation

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\,.$$

Using this one obtain the value of value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ as

(a) $2\pi^2$ (b) π^2 (c) $\frac{\pi^2}{3}$ (d) $\frac{\pi^2}{6}$

Q4. The Fourier series expansion of $x^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{l}\right)$

Then $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... =$

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi^2}{3}$ (d) $\frac{\pi^2}{12}$

Q5. For the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\cos nx + b_n \sin nx\right)$ of the function

$$f(x) = \begin{cases} 1 & -1 \le x < 0 \\ \frac{1}{2} & x = 0 \\ x & 0 < x \le 1 \end{cases} \quad \text{on } [-1,1]$$

The value of a_0 is

(a) $\frac{3}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) 1

Q6. For the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\cos nx + b_n \sin nx\right)$ of the function

$$f(x) = \begin{cases} -k & for -\pi < x < 0 \\ k & for\ 0 < x < \pi \end{cases}$$

then the value of $\frac{\pi}{4}$ is

(a) $1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - ....$ (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + ...$ (d) $1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - ....$

Q7. The Fourier coefficients b_n of the function $f(x)=\begin{cases}-1 & -\pi<x<\frac{-\pi}{2}\\ 0 & \frac{-\pi}{2}<x<\frac{\pi}{2}\\ 1 & \frac{\pi}{2}<x<\pi\end{cases}$ is,

(a) $\frac{2(-1)^n}{n}$ (b) $\frac{(-1)^n}{n\pi}$ (c) $\frac{2(-1)^n}{n\pi}$ (d) $\frac{-2(-1)^n}{n\pi}$

Q8. Find the Fourier Series for $f(x)=|\sin x|$ for $-\pi<x<\pi$.

(a) $\frac{1}{2}-\frac{1}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+.....\right]$ (b) $\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{3}\cos 2x+\frac{1}{15}\cos 4x+...\right]$

(c) $\frac{4k}{\pi}\left[\frac{1}{1}\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+....\right]$ (d) $-\frac{4}{\pi}\left[\frac{1}{3}\cos 2x+\frac{1}{15}\cos 4x+...\right]$

Q9. The function $f(x) = x^2$, $-\pi<x<\pi$ has a Fourier series representation

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \quad .$$

Using this one obtain the value of value of $\sum\frac{1}{(2n-1)^2}$ as

(a) $\frac{\pi^2}{6}$ (b) $\frac{\pi^2}{12}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{3}$

Q10. The Fourier coefficient b_n of the function $f(x)=x^3;\ -\pi<x<\pi$ is

(a) $\frac{\pi}{n}$ (b) $(-1)^n 2\left[\frac{6}{n^3}-\frac{\pi^2}{n}\right]$ (c) $\frac{2}{n}$ (d) 0

NAT (Numerical Answer Type)

Q11. Consider a periodic function $f(x)=\begin{cases}2 & -2<x<0\\ x & 0<x<2\end{cases}$. The Fourier coefficient a_0 is _____.

Q12. Consider a periodic function $f(x)=\frac{ax}{2\pi}$, where $0<x<2\pi$. The Fourier coefficient a_n is _____.

Q13. A periodic function is defined as $f(x)=1+x$ for the interval $0\le x\le 2$. The Fourier coefficient a_0 is _____.

Q14. Consider a periodic function $f(x)=x+1$ $0<x<\pi$ and Fourier series of f(x) is represented by $\frac{a_0}{2}+\sum(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$ then b_2 is _____

Q15. Fourier coefficient a_n in the Fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ of $f(x)=\frac{ax}{2\pi}$, where $0<x<2\pi$ is _____.

MSQ (Multiple Select Questions)

Q16. Consider the function $f(x)=\begin{cases}-1 & -\pi<x<\dfrac{-\pi}{2}\\ 0 & \dfrac{-\pi}{2}<x<\dfrac{\pi}{2}\\ 1 & \dfrac{\pi}{2}<x<\pi\end{cases}$.

which of the following is true for the Fourier coefficients

(a) $a_0=\pi$ (b) $a_n=0$ (c) $b_n=-\dfrac{2(-1)^n}{n\pi}$ (d) $b_n=\dfrac{2(-1)^n}{n\pi}$

Q17. Fourier series of $f(x)$ is represented by $\frac{a_0}{2}+\sum(a_n\cos nx+b_n\sin nx)$. Which of the following is true for the Fourier coefficients of period function $f(x)=\begin{cases}1 & 0<x<\dfrac{\pi}{2}\\ 0 & \dfrac{\pi}{2}<x<\pi\end{cases}$.

(a) $a_0=1$ (b) $a_n=\dfrac{2}{\pi n}$ (c) $b_n=\dfrac{2}{\pi n}$ (d) $b_n=0$

Q18. Fourier series expansion of $x^2=\dfrac{\pi^2}{3}+4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx$ then

(a) $\dfrac{\pi^2}{12}=\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots\ldots\ldots\ldots$ (b) $\dfrac{\pi^2}{6}=\left[\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\ldots.\right]$

(c) $\dfrac{\pi^2}{48}=\dfrac{1}{2^2}-\dfrac{1}{4^2}+\dfrac{1}{6^2}-\dfrac{1}{8^2}+\ldots\ldots\ldots\ldots$ (d) $\dfrac{\pi^2}{3}=\dfrac{1}{1^2}-\dfrac{1}{3^2}-\dfrac{1}{5^2}+\ldots\ldots\ldots\ldots$

Q19. Consider a periodic function $f(x)=\begin{cases}2 & -2<x<0\\ x & 0<x<2\end{cases}$. The Fourier series of f(x) is represented by $\frac{a_0}{2}+\sum(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$ then

(a) $a_0=3$ (b) $a_n=\frac{-4}{n\pi}$ (b) $b_n=\frac{-2}{n\pi}$ (b) $a_n=-\frac{4}{n^2\pi^2}$

Q20. If $f(x)=\begin{cases}1, & -1\le x<0\\ -2, & 0\le x<1\end{cases}$, and Fourier series of f(x) is represented by $\frac{a_0}{2}+\sum(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$ then which are true for Fourier coefficients?

(a) $a_0=0$ (b) $a_n=0$ (b) $a_0=-3$ (b) $b_n=\frac{2}{n\pi}$

Solution

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: Let $x+x^2=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+....+b_1\sin x+b_2\sin 2x+...$ ---(1)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx$$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)\cos nx\ \ dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\ \ dx$$

$$=\frac{1}{\pi}\left[\left(x+x^2\right)\frac{\sin nx}{n}-\left(2x+1\right)\left(-\frac{\cos nx}{n^2}+\left(2\right)\left(-\frac{\sin nx}{n^3}\right)\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\left(2\pi+1\right)\frac{\cos n\pi}{n^2}-\left(-2\pi+1\right)\frac{\cos\left(-n\pi\right)}{n^2}\right]=\frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right]=\frac{4\left(-1\right)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\left(x+x^2\right)\left(-\frac{\cos nx}{n}\right)-\left(2x+1\right)\left(-\frac{\sin nx}{n^2}\right)+2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[-\left(\pi+\pi^2\right)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}+\left(-\pi+\pi^2\right)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]=-\frac{2}{n}\left(-1\right)^n$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$x+x^2=\frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+....\right]$$

$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+....\right] \quad ---(2)$$

Put $x=\pi$ in (2), $\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$ ---(3)

Put $x=-\pi$, $-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$ ---(4)

Adding (3) and (4) $2\pi^2=\frac{2\pi^2}{3}+8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]$

$$\frac{4\pi^2}{3}=8\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]\Rightarrow\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

.

Ans. 2: (a)

Solution: $a_0=\frac{1}{\pi}\int_{-\pi}^{0}(-k)dx+\int_0^{\pi}kdx=\frac{1}{\pi}\left[-k(\pi)+k(\pi)\right]=0$, $a_n=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin nxdx+\frac{1}{\pi}\int_0^{\pi}k\sin nxdx=\frac{-k}{\pi}\frac{(-\cos nx)}{n}\Big|_{-\pi}^{0}+\frac{k}{\pi}\frac{(-\cos nx)}{n}\Big|_0^{\pi}$$

$$\frac{k}{\pi}\left(\frac{1-(-1)^n}{n}\right)-\frac{k}{\pi}\left(\frac{(-1)^n-1}{n}\right)=\frac{2k}{n\pi}\left(1-(-1)^n\right)$$

$$f(x)=\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n}\sin nx=\frac{2k}{\pi}\left[\frac{2}{1}\sin x+\frac{2}{3}\sin 3x+\frac{2}{5}\sin 5x+.....\right]$$

$$f(x)=\frac{4k}{\pi}\left[\frac{1}{1}\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+....\right]$$

Ans. 3: (d)

Solution: $f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

$$x^2=\frac{\pi^2}{3}+4\left[\frac{-1}{12}\cos x+\frac{1}{2^2}\cos 2x+\frac{-1}{3^2}\cos 3x+\frac{1}{4^2}\cos 4x+....\right]$$

$$x^2=\frac{\pi^2}{3}+4\left[\frac{-1}{1^2}\cos x+\frac{1}{2^2}\cos 2x+\frac{-1}{3^2}\cos 3x+\frac{1}{4^2}\cos 4x-...\right]$$

at $x=\pi$, $\pi^2-\frac{\pi^2}{3}=4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]\Rightarrow\frac{2\pi^2}{3.4}=\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

Ans. 4: (d)

Solution: The given Fourier series expansion is

$$x^2=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

Substitute $x=0$ in above equation $0=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 0$

$$\Rightarrow 0=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow 0=\frac{1}{3}+\frac{4}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-.....\right]$$

$$\Rightarrow=\frac{1}{3}+\frac{4}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-.....\right]\Rightarrow\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.....=\frac{\pi^2}{12}$$

Ans. 5: (a)

Solution: The general Fourier series on $[-L,L]$ is

$$f(x)=\frac{a_0}{2}+\sum_{n-1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\qquad\text{where } a_0=\int_{-L}^{+L}f(x)\,dx$$

$$\therefore a_0 = \frac{1}{L}\int_{-1}^{+1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = \int_{-1}^{0} 1dx + \int_{0}^{1} xdx = \frac{3}{2}$$

Ans. 6: (b)

Solution: $a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi} kdx = \frac{1}{\pi}[-k(\pi)+k(\pi)] = 0, \qquad a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin nxdx + \frac{1}{\pi}\int_{0}^{\pi} k\sin nxdx = \frac{-k}{\pi}\frac{(-\cos nx)}{n}\Big|_{-\pi}^{0} + \frac{k}{\pi}\frac{(-\cos nx)}{n}\Big|_{0}^{\pi}$$

$$\frac{k}{\pi}\left(\frac{1-(-1)^n}{n}\right) - \frac{k}{\pi}\left(\frac{(-1)^n-1}{n}\right) = \frac{2k}{n\pi}\left(1-(-1)^n\right)$$

$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)}{n}\sin nx = \frac{2k}{\pi}\left[\frac{2}{1}\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + ...\right]$$

$$f(x) = \frac{4k}{\pi}\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$$

$$f(x)\Big|_{x=\frac{\pi}{2}} = k$$

$$k = \frac{4k}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....\right] \Rightarrow \frac{\pi}{4} = \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}.....\right)$$

Ans. 7: (d)

Solution: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}}(-1)dx + \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 0dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(1)dx \Rightarrow a_0 = \frac{1}{\pi}\left[-\left(\frac{-\pi}{2}+\pi\right)+\left(\pi-\frac{\pi}{2}\right)\right] = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}} -\cos nxdx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos nxdx \Rightarrow a_n = \frac{-1}{\pi}\frac{\sin nx}{n}\Big|_{-\pi}^{\frac{-\pi}{2}} + \frac{1}{\pi}\frac{\sin nx}{n}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$a_n = \frac{-1}{\pi}\left(\frac{-1}{n}\right) + \frac{1}{\pi}\frac{(-1)}{n} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}} -\sin nx dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin nx dx \Rightarrow b_n = \frac{-1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{\frac{-\pi}{2}} + \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$b_n = \frac{1}{n\pi}\left[-(-1)^n\right] - \frac{1}{n\pi}(-1)^n = \frac{-2(-1)^n}{n\pi}$$

Ans. 8: (b)

Solution: $a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x dx + \frac{1}{\pi}\int_{0}^{\pi}\sin x dx = \frac{-1}{\pi}\left[-\cos x\Big|_{-\pi}^{0}\right] + \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{4}{\pi}, \quad b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin x\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}\sin x\cos nx dx$$

$$= \frac{-1}{2\pi}\int_{-\pi}^{0}\left[\sin(1+n)x + \sin(1-n)x\right]dx + \frac{1}{2\pi}\int_{0}^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{-1}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)x}{(1-n)}\Big|_{-\pi}^{0}\right] + \frac{1}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)x}{(1-n)}\Big|_{0}^{\pi}\right]$$

$$= \frac{+1}{2\pi}\left[\frac{\cos(1+n)x}{(1+n)} + \frac{\cos(1-n)x}{(1-n)}\Big|_{-\pi}^{0} - \left\{\frac{\cos(1+n)x}{(1+n)} + \frac{\cos(1-n)x}{1-n}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n} - \left\{\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1-n}}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2}{1+n} + \frac{2}{1-n} - \frac{2(-1)^{1-n}}{1+n} - \frac{2(-1)^{1-n}}{1-n}\right] = \frac{1}{\pi}\left[\frac{1-(-1)^{1+n}}{1+n} + \frac{1-(-1)^{1-n}}{1-n}\right]$$

$n = 1, 3, 5, 9, \ldots.$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^{1+n} - n + n(-1)^{1+n} + 1 - (-1)^{1-n} + n - n(-1)^{1-n}}{1-n^2}\right]$$

$$= \frac{1}{\pi(1-n^2)}\left[2+(-1)^{1+n}(n-1)-(-1)^{1-n}(n+1)\right]$$

$n=1$ not exist

$$n=2 = \frac{1}{\pi(1-4)}\left[2+-(1)+3\right] = \frac{-4}{3\pi}, \quad n=3\ \frac{1}{\pi(1-9)}\left[2+2-4\right]=0,$$

$$n=4 = \frac{1}{\pi(1-16)}\left[2+-(3)+(5)\right] = \frac{-4}{15\pi}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \ldots\right]$$

Ans. 9: (b)

Solution: $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

$$x^2 = \frac{\pi^2}{3} + 4\left[\frac{-1}{12}\cos x + \frac{1}{2^2}\cos 2x + \frac{-1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x + \ldots.\right]$$

$$x^2 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{-1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x - \ldots\right]$$

at $x=0$

$$\frac{-\pi^2}{12} = \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2}\ldots\ldots\right]$$

$$\Rightarrow \frac{\pi^2}{12} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots.\right) = \Sigma\frac{1}{(2n-1)^2}$$

Ans. 10: (b)

Solution: $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx \quad dx = \frac{x^3(-\cos nx)}{\pi n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 3x^2\frac{(-\cos nx)}{\pi n}dx$

$$= \frac{-1}{n}\left[\pi^3(-1)^n + \pi^3(-1)^n\right] + \frac{3}{n}\int_{-\pi}^{\pi} x^2\cos nx dx$$

$$= \frac{1}{\pi}\left[\frac{-2\pi^3(-1)^n}{n}\right] + \frac{1}{\pi}\left[\frac{12\pi(-1)^n}{n^3}\right] = \left[\frac{12(-1)^n}{n^3} - \frac{\pi^2 2(-1)^n}{n}\right]$$

$$= (-1)^n 2\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right]$$

Ans. 11: 3

Solution: $a_0 = \frac{1}{2}\int_{-2}^{0} 2dx + \frac{1}{2}\int_{0}^{2} xdx = 2 + \frac{1}{2}\frac{4}{2} = 3$

Ans. 12: 0

Solution: $a_n = \frac{1}{\pi}\frac{a}{2\pi}\int_0^{2\pi} x\cos nxdx = \frac{a}{2\pi^2}\left[\frac{x\sin nx}{n}\Big|_0^{2\pi} + \frac{\cos nx}{n^2}\Big|_0^{2\pi}\right] = \frac{a}{2\pi^2}[0] = 0$

Ans. 13: 4

Solution: $a_0 = \frac{2}{2}\int_0^2 (1+x)dx = 2 + \frac{4}{2} = 4$

Ans. 14: −1

Solution: $b_n = \frac{2}{\pi}\left[x\frac{(-\cos nx)}{n}\Big|_0^{\pi} + \frac{\sin nx}{n^2}\Big|_0^{\pi}\right] + \frac{2}{\pi}\frac{(-\cos nx)}{n}\Big|_0^{\pi}$

$$= -\frac{2}{\pi}\frac{\pi}{n}(-1)^n + \frac{2}{\pi}\left(\frac{-1}{n}\right)\left[(-1)^n - 1\right] = -(-1)^n\frac{2}{n} - \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{2}{n}\left[1 + \frac{2}{\pi}\right] \qquad n \text{ is odd}$$

$$b_n = -\frac{2}{n} \qquad n \text{ is even}$$

Ans. 15: 0

Solution: $a_n = \frac{1}{\pi}\frac{a}{2\pi}\int_0^{2\pi} x\cos nx dx = \frac{a}{2\pi^2}\left[\frac{x\sin nx}{n}\Big|_0^{2\pi} + \frac{\cos nx}{n^2}\Big|_0^{2\pi}\right] = \frac{a}{2\pi^2}[0] = 0$

Ans. 16: (b) and (c)

Solution: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}}(-1)dx + \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}0dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(1)dx \Rightarrow a_0 = \frac{1}{\pi}\left[-\left(\frac{-\pi}{2}+\pi\right)+\left(\pi-\frac{\pi}{2}\right)\right] = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}} -\cos nx dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos nx dx \Rightarrow a_n = \frac{-1}{\pi}\frac{\sin nx}{n}\Big|_{-\pi}^{\frac{-\pi}{2}} + \frac{1}{\pi}\frac{\sin nx}{n}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$a_n = \frac{-1}{\pi}\left(\frac{-1}{n}\right) + \frac{1}{\pi}\frac{(-1)}{n} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\frac{-\pi}{2}} -\sin nx dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin nx dx \Rightarrow b_n = \frac{-1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{\frac{-\pi}{2}} + \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$b_n = \frac{1}{n\pi}\left[-(-1)^n\right] - \frac{1}{n\pi}(-1)^n = \frac{-2(-1)^n}{n\pi}$$

Ans. 17: (a), (b) and (c)

Solution: $a_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1dx = 1$

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos nx dx = \frac{2}{\pi}\frac{\sin nx}{n}\Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi n} \text{ for } n \text{ is odd}$$

$$b_n = -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi n}$$

Ans. 18: (a), (b) and (c)

Solution: $f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

$$x^2=\frac{\pi^2}{3}+4\left[\frac{-1}{12}\cos x+\frac{1}{2^2}\cos 2x+\frac{-1}{3^2}\cos 3x+\frac{1}{4^2}\cos 4x+....\right]$$

$$x^2=\frac{\pi^2}{3}+4\left[\frac{-1}{1^2}\cos x+\frac{1}{2^2}\cos 2x+\frac{-1}{3^2}\cos 3x+\frac{1}{4^2}\cos 4x-...\right]$$

For $x=\pi$

$$\pi^2-\frac{\pi^2}{3}=4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\right]\Rightarrow\frac{2\pi^2}{3.4}=\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

For $x=0$, $\frac{-\pi^2}{12}=\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}.....\right]\Rightarrow\frac{\pi^2}{12}=\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}.....\right)$

For $x=\pi/2$, $\frac{\pi^2}{4}-\frac{\pi^2}{3}=4\left[-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{6^2}+\frac{1}{8^2}+....\right]$

$$\frac{\pi^2}{48}=\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-\frac{1}{8^2}+.....$$

Ans. 19: (a), (c) and (d)

Solution: $a_0=\frac{1}{2}\int_{-2}^{0}2dx+\frac{1}{2}\int_{0}^{2}xdx=2+\frac{1}{2}\frac{4}{2}=3$

$$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\left(\frac{n\pi x}{2}\right)dx=\frac{1}{2}\int_{-2}^{0}2\cos\left(\frac{n\pi x}{2}\right)dx+\frac{1}{2}\int_{0}^{2}x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$=\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\Bigg|_{-2}^{0}+\frac{1}{2}\frac{x\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\Bigg|_{0}^{2}+\frac{1}{2}\frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\Bigg|_{0}^{2}=\frac{1}{2}\left[(-1)^n-1\right]\frac{4}{n^2\pi^2}=\frac{-4}{n^2\pi^2}\qquad\text{for odd } n$$

$$b_n = \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi}{2}\,dx = \left.\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right|_{-2}^{0} + \frac{1}{2}\left.\frac{x\left(-\cos\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right. + \frac{1}{2}\left.\frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right|_{0}^{2}$$

$$= \frac{-2}{n\pi}\left[1-(-1)^n\right] + \frac{1}{n\pi}(-2)(-1)^n = \frac{-2}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2(-1)^n}{n\pi} = \frac{-2}{n\pi}$$

Ans. 20: (b), (c) and (d)

Solution: $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1dx + \int_{0}^{1} -2dx = -3$

$$a_n = \int_{-1}^{0} 1\cos n\pi x dx + \int_{0}^{1} -2\cos n\pi x dx = \left.\sin\frac{n\pi x}{n\pi}\right|_{-1}^{0} + \left.\frac{2\sin n\pi x}{n\pi}\right|_{0}^{1} = \frac{1}{n\pi}[0-0] + \frac{2}{n\pi}[0] = 0$$

$$b_n = \left.\frac{-\cos n\pi x}{n\pi}\right|_{-1}^{0} + \left.\frac{-2\cos n\pi}{n\pi}\right|_{0}^{1} = \frac{-1}{n\pi}\left[1-(-1)^n\right] - \frac{2}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{1}{n\pi}\left[(-1)^n - 1\right] - \frac{2}{n\pi}\left[(-1)^n - 1\right] = \frac{-1}{n\pi}\left[(-1)^n - 1\right] = \frac{2}{n\pi}$$

4. Calculus of Single and Multiple Variables

4.1 Limits

Definition:

The number A is said to be the limit of $f(x)$ at $x=a$ if for any arbitrary chosen positive number ε, however small but not zero, there exists a corresponding number δ greater than zero such that $|f(x)-A|<\varepsilon$

For all values of x for which $0<|x-a|<\delta$.

4.1.1 Right Hand and Left hand Limits

If x approaches a from larger values of x than a, then $\lim_{x\to a+0} f(x)$ or $f(a+0)$.

"Put $a+h$ for x in $f(x)$ and make h approaches zero". In short, we have

$f(a+0)=\lim_{h\to 0} f(a+h)$.

If x approaches a from smaller values of x than a, then $\lim_{x\to a-0} f(x)$ or $f(a-0)$.

In this case $f(a-0)=\lim_{h\to 0} f(a-h)$

If both right hand and left hand limits of $f(x)$, as $x\to a$, exist and are equal in value, their common value, evidently, will be the limit of $f(x)$, as $x\to a$.

4.1.2 Theorem of Limits

If $f(x)$ and $g(x)$ are two functions then

1. $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$ 2. $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

3. $\lim_{x\to a}[f(x).g(x)]=\left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$ 4. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

5. $\lim_{x\to a}[kf(x)]=k\left(\lim_{x\to a} f(x)\right)$ where k is constant

6. $\lim_{x\to a}\sqrt{f(x)}=\sqrt{\lim_{x\to a} f(x)}$

7. $\lim_{x\to a}\left[f(x)\right]^{p/q} = \left[\lim_{x\to a} f(x)\right]^{p/q}$: where p & q are integers

Some Important Expansions

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - .....$
2. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .....$
4. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$
5. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$
6. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x3}{3!} + ...$
7. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x3}{3!} + ...$
8. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}....., \quad \text{if } |x| < 1$
9. $\log_e(1-x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}.....\right\}$

4.1.3 L'Hospital's Rule

If a function $f(x)$ takes the form $\frac{0}{0}$ at $x = a$, then we say that $f(x)$ is indeterminate at $x = a$.

If $\phi(x)$ and $\psi(x)$ are functions of x such that $\phi(a) = 0$ and $\psi(a) = 0$, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}.$$

Some Important Standard Limits

1. $\lim_{x\to a} x = a$
2. $\lim_{x\to a} c = c$: where c is constant $c \in R$
3. $\lim_{x\to a} x^n = a^n; \quad n \in R$
4. $\lim_{x\to a}\frac{x^n - a^n}{x - a} = na^{n-1}; \quad n \in N, a > 0$
5. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$
6. $\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1$
7. $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$
8. $\lim_{\theta\to 0}\frac{\theta}{\tan\theta} = 1$
9. $\lim_{\theta\to 0}\frac{\sin k\theta}{\theta} = k$
10. $\lim_{\theta\to 0}\frac{\tan k\theta}{\theta} = k$

11. $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

12. $\lim_{x\to 0}\frac{x^2}{1-\cos x}=2$

13. $\lim_{x\to 0}\frac{\sin mx^0}{\sin nx^0}=\frac{m}{n}$

14. $\lim_{x\to 0}\frac{1-\cos kx}{x^2}=\frac{k^2}{2}$

15. $\lim_{x\to 0}\cos x=1;\quad \lim_{x\to \pi/2}\cos x=0$

16. $\lim_{x\to 0}\sin x=0;\quad \lim_{x\to \pi/2}\sin \pi/2=1$

17. $\lim_{x\to a}\frac{a^x-1}{x}=\log a$ where $a>0$

18. $\lim_{x\to a}\frac{e^x-1}{x}=1$

19. $\lim_{x\to a}\frac{\log(1+x)}{x}=1;\quad \lim_{x\to a}\frac{\log_a(1+x)}{x}=\log a^e \quad a>0$

20. $\lim_{x\to 0}(1+x)^{1/x}=e;\quad \lim_{x\to 0}(1+kx)^{1/x}=e^k$

21. $\lim_{x\to 0}\frac{\log(1+kx)}{x}=k$

22. $\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}=\frac{a^2-b^2}{c^2-d^2}$

23. $\lim_{x\to 0}\left(\frac{\cos ax-\cos bx}{x}\right)=\frac{b^2-a^2}{2}$

24. $\lim_{x\to 0}\left(\frac{1+ax}{1+bx}\right)^{1/x}=e^{a-b}$

25. $\lim_{x\to 0}\left(\frac{a+bx}{a+cx}\right)^{1/x}=e^{\frac{b-c}{a}}$

26. $\lim_{x\to \infty}\frac{1}{x}=0;\ \lim_{x\to -\infty}\frac{1}{x}=0$

27. $\lim_{x\to \infty}\frac{1}{x^2}=0;\ \lim_{x\to -\infty}\frac{1}{x^2}=0$

28. $\lim_{x\to \infty}\frac{1}{x^k}=0$ where $k>0$

29. $\lim_{x\to \infty}k=k;\quad \lim_{x\to -\infty}k=k$ where k is constant

30. $\lim_{x\to a}\sin x=\sin a$

31. $\lim_{x\to a}\cos x=\cos a$

32. $\lim_{x\to 1}\frac{x+x^2+x^3+\ldots x^n-n}{x-1}=\frac{n(n+1)}{2}$

33. $\lim_{x\to 0}\frac{a^x-b^x}{x}=\log\left(\frac{a}{b}\right);\quad a,b>0$

34. $\lim_{x\to 0}\frac{\sec x-1}{x^2}=\frac{1}{2};\lim_{x\to 0}\frac{\operatorname{cosec}x-1}{x^2}=1$

35. $\lim_{x\to \infty}\left(1+\frac{1}{x^2}\right)^x=e^x;\lim_{n\to \infty}\left(1+\frac{1}{n}\right)^n=e$

36. $\lim_{h\to \infty}\left(1+\frac{a}{h}\right)^h=e^a$

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4.1.4 Continuity

Definition

$f(x)$ is said to be continuous at $x=a$ if $R=L=V$ i.e.,

$$\lim_{h\to 0} f(a+h)=\lim_{h\to 0} f(a-h)=f(a)$$

In case a function is not defined at $x=a$, i.e. $f(a)$ does not exist or in case the $R.H.L \neq L.H.L$, then we say that the function is discontinuous at $x=a$. Its graph will show a break at $x=a$.

Example:

(i) $f(x)=\dfrac{x^2-a^2}{x-a}$ is discontinuous at $x=a$ as $f(a)$ does not exist.

(ii) $f(x)=\dfrac{1}{1+2^{1/x}}$ is discontinuous at $x=0$ because

$$R.H.L=\lim_{h\to 0}\frac{1}{1+2^{1/h}}=0 \text{ and } L.H.L=\lim_{h\to 0}\frac{1}{1+2^{1/-h}}=1 \Rightarrow R.H.L \neq L.H.L$$

(iii) $f(x)=1-\cos e^{1/x}$ is discontinuous at $x=0$ as $f(0)=1-\cos e^{1/0}$ is undefined. It is oscillating.

Example: Find the limits $\lim_{x\to 0}\dfrac{1-cox}{x^2}$

Solution: $$\lim_{x\to 0}\frac{1-cox}{x^2}\Rightarrow\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2=\frac{1}{2}$$

Example: Find the limit $\lim_{x\to\infty}\dfrac{\ln(1+x)}{3^x-1}$

Solution: $$\Rightarrow\lim_{x\to\infty}\left[\frac{\ln(1+x)}{x}.\frac{x}{3^x-1}\right]=\frac{1}{\ln 3}$$

Example: Find the limit $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x \Rightarrow \lim_{x\to\infty}\left[\left(1+\frac{1}{x^2}\right)^{x^2}\right]^{\frac{1}{x}} = e^0 = 1$$

Example: Let the function $f(x)$ be defined by

$$f(x)=\begin{Bmatrix} e^x, & x \text{ is rational} \\ e^{1-x}, & x \text{ is irrational} \end{Bmatrix}$$

for x in $(0,1)$. Then

(a) f is continuous at every point in $(0,1)$

(b) f is discontinuous at every point in $(0,1)$

(c) f is discontinuous only at one point in $(0,1)$

(d) f is continuous only at one point in $(0,1)$

Ans. (d)

Solution: $f(x)$ will be continuous at those points only where rational and irrational values coincide

i.e. $e^x = e^{1-x} \Rightarrow e^x = \frac{e}{e^x}$. So $f(x)$ is continuous at $x = 1/2$ only.

Example Find the value of $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}}$ is

Solution: $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k}{n}}}$

$$= \int_0^1 \frac{1}{\sqrt{1+x}}dx = \left.\frac{\sqrt{1+x}}{1/2}\right|_0^1 = 2\left(\sqrt{2}-1\right)$$

4.2 Differentiability

Definition

$f(x)$ is said to be differentiable at $x=a$ if $R'=L'$ i.e,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

Example: If $f(x)=e^{-x^2}$ and $g(x)=|x|e^{-x^2}$, then

(a) f and g are differentiable everywhere

(b) f is differentiable everywhere but g is not

(c) g is differentiable everywhere but f is not

(d) g is discontinuous at $x=0$

Ans: (b)

Solution: $f(x)=e^{-x^2}$ is differentiable but $g(x)=|x|e^{-x^2}$ is not differentiable.

$$g(x)=\begin{cases}-xe^{-x^2}; & x<0\\ xe^{-x^2}; & x>0\end{cases}$$

Left hand Limit $\lim_{h\to 0} g(x-h)=-(x-h)e^{-(x-h)^2}$

Right hand Limit $\lim_{h\to 0} g(x+h)=(x+h)e^{-(x+h)^2} \Rightarrow \lim_{h\to 0} g(x-h)\neq \lim_{h\to o} g(x+h)$

Example: Differentiate $y=\frac{x}{1+x}$

Solution: $\frac{dy}{dx}=\frac{(1+x)\frac{dx}{dx}-x\frac{d(1+x)}{dx}}{(1+x)^2}=\frac{1}{(1+x)^2}$

Example: Differentiate $y=\sqrt{\sin\sqrt{x}}$

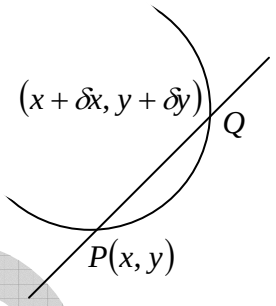
Solution: $\frac{d\sqrt{\sin\sqrt{x}}}{dx}=\frac{d\sqrt{\sin\sqrt{x}}}{d\sin\sqrt{x}}.\frac{d\sin\sqrt{x}}{d\sqrt{x}}.\frac{d\sqrt{x}}{dx} \Rightarrow \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\cos\sqrt{x}\frac{1}{2}x^{-1/2}$

4.2.1 Tangents and Normal

Tangent at (x, y) to $y = f(x)$

Let $y = f(x)$ be a given curve and $P(x, y)$and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on it. Equation of the line PQ is

$$Y - y = \frac{y + \delta y - y}{x + \delta x - x}(X - x) \text{ or } Y - y = \frac{\delta y}{\delta x}(X - x)$$



This line will be tangent to the given curve at P if $Q \to P$ which in tern means that $\delta x \to 0$ and we know that $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

Therefore the equation of the tangent is $Y - y = \frac{dy}{dx}(X - x)$

Normal at (x, y)

The normal at (x, y) being perpendicular to tangent will have its slope as $-1 \Big/ \frac{dy}{dx}$ and hence its equation is $Y - y = \frac{-1}{dy/dx}(X - x)$

Geometrical meaning of dy/dx

dy/dx represents the slope of the tangent to the given curve $y = f(x)$ at any point (x, y)

$$\therefore \frac{dy}{dx} = \tan\psi$$

where ψ is the angle which the tangent to the curve makes with $+ve$ direction of x-axis. In case we are to find the tangent at any point (x_1, y_1) then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ *i.e.* the value of $\frac{dy}{dx}$ at (x_1, y_1) will represent the slope of the tangent and hence its equation in this case will be

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$

Normal $y - y_1 = \dfrac{-1}{(dy/dx)_{(x_1, y_1)}}(x - x_1)$

4.2.2 Condition for tangent to be parallel or perpendicular to *x*-axis

If tangent is parallel to *x*-axis or normal is perpendicular to *x*-axis then $\dfrac{dy}{dx} = 0$

If tangent is perpendicular to *x*-axis or normal is parallel to *x*-axis then

$\dfrac{dy}{dx} = \infty$ or $\dfrac{dx}{dy} = 0$

4.2.3 Maxima and Minima

For the function $y = f(x)$ at the maximum as well as minimum point the tangent is parallel to *x*-axis so that its slope is zero.

Calculate $\dfrac{dy}{dx} = 0$ and solve for *x*. Suppose one root of $\dfrac{dy}{dx} = 0$ is at $x = a$.

If $\dfrac{d^2y}{d^2x} = -ve$ for $x = a$, then maximum at $x = a$.

If $\dfrac{d^2y}{d^2x} = +ve$ for $x = a$, then minimum at $x = a$.

If $\dfrac{d^2y}{d^2x} = 0$ at $x = a$, then find $\dfrac{d^3y}{d^3x}$.

If $\dfrac{d^3y}{d^3x} \neq 0$ at $x = a$, neither maximum nor minimum at $x = a$.

If $\dfrac{d^3y}{d^3x} = 0$ at $x = a$, then find $\dfrac{d^4y}{d^4x}$.

If $\dfrac{d^4y}{d^4x} > 0$ i.e $+ve$ at $x = a$, then *y* is minimum at $x = a$ and if $\dfrac{d^4y}{d^4x} < 0$ i.e *-ve* at $x = a$, then *y* is maximum at $x = a$ and so on.

4.3 Partial Differentiation

If a derivative of function of several independent variable be found with respect to any one of them, keeping the others as constants it is said to be partial derivative .the operation of finding the partial derivative of function of more than one independent variable is called partial differentiation.

If $u = f(x, y)$ then $\frac{\partial u}{\partial x} \equiv f_x$ and $\frac{\partial u}{\partial y} \equiv f_y$

We call $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \equiv \frac{\partial^2 u}{\partial x^2} \equiv f_{xx}$ and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \equiv \frac{\partial^2 u}{\partial y^2} \equiv f_{yy}$,

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \equiv \frac{\partial^2 u}{\partial x \partial y} \equiv f_{xy}$ and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \equiv \frac{\partial^2 u}{\partial y \partial x} \equiv f_{yx}$ it is given that $f_{xy} = f_{yx}$

4.3.1 Euler Theorem of Homogeneous Function

If u is homogeneous function of $x,$ and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

For n variable function $f = f\left(x_1, x_2, x_3 \ldots\ldots x_n\right)$ of degree n then $x_1\frac{\partial f}{\partial x} + x_2\frac{\partial f}{\partial x_2} \ldots\ldots x_n\frac{\partial f}{\partial x_n} = nu$

Total derivative $u = f(x, y)$ $x = \phi(t)$ and $y = \psi(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

Differentiation of implicit functions $f(x, y) = c$ be an implicit relation between x and y which defines as a differential function of x then

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Change in variable $u = f(x, y)$ $x = \phi(s,t)$ $y = \phi(s,t)$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial s}$ and $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t}$

4.3.2 Maxima and Minima (of function of two independent variables)

The necessary condition that $f(x,y)$ should have maximum or minimum at $x=a, y=b$ is that

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}}=0, \text{ and } \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}}=0$$

Sufficient condition for maxima and minima

Let $r=\left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}$ $s=\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{\substack{x=a\\y=b}}$ $t=\left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$

Case 1-if $f(x,y)$ will have maximum or minimum at $x=a, y=b$ if $rt>s^2$ further $f(x,y)$ is maximum or minimum according as r in negative or positive.

Case 2-if $f(x,y)$ will have maximum or minimum at $x=a, y=b$ if $rt<s^2$ further $f(x,y)$ is saddle point .

Case 3 – if $rt=s^2$ this case is doubtful case and further advanced investigation is needed to determined whether $f(x,y)$ a maximum or minimum at or not .

Example: For what values of x and y, does the integral $\int_x^y\left(6-t-t^2\right)dt$ attain its maximum?

Solution: $I(x,y)\,or\,f(x,y)=\int_x^y\left(6-t-t^2\right)dt$

$f_x=-\left(6-x-x^2\right)=x^2+x-6=(x+3)(x-2)$, $f_y=6-y-y^2=-(y+3)(y-2)$

$f_x=0\Rightarrow x=2,-3$, $f_y=0\Rightarrow y=2,-3$

so, stationary points are $(2,2),(2,-3),(-3,2)\&(-3,-3)$

$f_{xx}=2x+1$, $f_{yy}=-2y-1$, $f_{xy}=0$

$f_{xx}f_{yy}-\left(f_{xy}\right)^2=-(2x+1)(2y+1)$

At $x=-3$ and $y=2$ and $f_{xx}f_{yy}-\left(f_{xy}\right)^2>0$

so maximum value of $f(x,y)$ is obtained at $x=-3$ and $y=2$

Example: If $z = z(x, y)$ then prove that $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

Solution: $dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$, $dy = \left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz$

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x \left\{\left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz\right\}$$

$$I = \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial z}\right)_y \text{ and } \left(\frac{\partial z}{\partial x}\right)_y + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z = 0$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

Example: Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$ and prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$

Solution: We have $z = x^3 + y^3 - 3axy$

$\therefore$ $\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay$, and $\frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$

Also $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x$, and $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6x$ and $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a$

We observe that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$

Example: If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$,

Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution: We have $\frac{\partial u}{\partial y} = x^2 \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} - \left\{2y\tan^{-1}\frac{x}{y} + y^2 \cdot \frac{1}{1+(x/y)^2}\left(\frac{-x}{y}\right)\right\}$

$$= \frac{x^2}{x^3+y^3} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = x - 2y\tan^{-1}\frac{x}{y}$$

$\therefore$ $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{x - 2y\tan^{-1}\frac{x}{y}\right\} = 1 - 2y\frac{1}{1+(x/y)^2}\cdot\frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

Similarly, $\frac{\partial u}{\partial x} = 2x\tan^{-1}\frac{y}{x} - y$

and $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{2x\tan^{-1}\frac{y}{x} - y\right\} = \frac{x^2-y^2}{x^2+y^2}$. Hence the result

Example: If $z = f(x+ct) + \phi(x-ct)$, prove that

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Solution: We have

$$\frac{\partial z}{\partial x} = f'(x+ct)\cdot\frac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$$

and $\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct)$ (i)

Again $\frac{\partial z}{\partial t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct) = cf'(x-ct) - c\phi'(x-ct)$

and $\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2\left[f''(x+ct) + \phi''(x-ct)\right]$ (ii)

From (i) and (ii), it follows that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

This is an important partial differential equation, known as wave equation.

Example: If $u=\log\left(x^3+y^3+z^3-3xyz\right)$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{\left(x+y+z\right)^2}$

Solution: we have $\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}, \frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}, \frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{x^3+y^3+z^3-3xyz}$

$=\frac{3\left(x^2+y^2+z^2-xy-yz-zx\right)}{\left(x+y+z\right)\left(x^2+y^2+z^2-xy-yz-zx\right)}=\frac{3}{x+y+z}$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)=\left(\frac{3}{x+y+z}\right)$

$=-\frac{3}{\left(x+y+z\right)^2}-\frac{3}{\left(x+y+z\right)^2}-\frac{3}{\left(x+y+z\right)}=-\frac{9}{\left(x+y+z\right)^2}$

Example: If $u=f\left(r\right)$ and $x=r\cos\theta, y=r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''\left(r\right)+\frac{1}{r}f'\left(r\right)$$

Solution: we have $\frac{\partial u}{\partial x}=f'\left(r\right).\frac{\partial r}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}=f''\left(r\right).\left(\frac{\partial r}{\partial x}\right)^2+f'\left(r\right).\frac{\partial^2 r}{\partial x^2}$

Similarly, $\frac{\partial^2 u}{\partial y^2}=f''\left(r\right).\left(\frac{\partial r}{\partial y}\right)^2+f'\left(r\right).\frac{\partial^2 r}{\partial y^2}$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''\left(r\right).\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]+f'\left(r\right)\left[\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right]$

now to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$ etc, we write $r=\left(x^2+y^2\right)^{\frac{1}{2}}$

$\frac{\partial r}{\partial x}=\frac{1}{2}\left(x^2+y^2\right)^{\frac{-1}{2}}.2x=\frac{x}{r}$ and $\frac{\partial^2 r}{\partial x^2}=\frac{r.1-x.\frac{\partial r}{\partial x}}{r^2}=\frac{r-\frac{x^2}{r}}{r^2}=\frac{y^2}{r^3}$

Similarly, $\frac{\partial r}{\partial y}=\frac{y}{x}$ and $\frac{\partial^2 r}{\partial y^2}=\frac{x^2}{r^3}$

Substituting the values of $\frac{\partial r}{\partial x}$ etc. in (i), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]+f'(r)\left[\frac{y^2}{r^3}+\frac{x^2}{r^3}\right]=f''(r)+\frac{1}{r}f'(r)$$

Example: Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$ where $\log u=\frac{\left(x^3+y^3\right)}{\left(3x+4y\right)}$

Solution: Since $z=\log u=\frac{x^3+y^3}{3x+4y}=x^2.\frac{1+\left(\frac{y}{x}\right)^3}{3+4\left(\frac{y}{x}\right)}$

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad \text{(i)}$$

But $\frac{\partial z}{\partial x}=\frac{1}{u}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\frac{1}{u}\frac{\partial u}{\partial y}$

Hence (i) becomes

$$x.\frac{1}{u}\frac{\partial u}{\partial x}+y.\frac{1}{u}\frac{\partial u}{\partial y}=2\log u \quad \text{or} \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$$

Example: If z is a homogeneous function of degree n in x and y, show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n\left(n-1\right)z$$

Solution: By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ (i)

Differentiating (i) partially w.r.t. x, we get $x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$

$$x\frac{\partial^2 z}{\partial y^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial y} \qquad \text{(ii)}$$

Again differentiating (i) partially w.r.t. y, we get $x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \qquad \text{(iii)}$$

Multiplying (ii) by x and (iii) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)z \qquad \text{(By (i))}$$

Example: Given $u=\sin\left(\frac{x}{y}\right), x=e^t$ and $y=t^2$, find $\frac{du}{dt}$ as a function of t. verify your result by direct substitution.

Solution: We have $\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}=\left(\cos\frac{x}{y}\right)\frac{1}{y}.e^t+\left(\cos\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)2t$

$$=\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^2}-2\cos\left(\frac{e^t}{t^2}\right).\frac{e^t}{t^3}=\left\{\frac{(t-2)}{t^3}\right\}e^t\cos\left(\frac{e^t}{t^2}\right)$$

Also $u=\sin\left(\frac{x}{y}\right)=\sin\left(\frac{e^t}{t^2}\right)$

$$\therefore \qquad \frac{du}{dt}=\cos\left(\frac{e^t}{t^2}\right).\frac{t^2e^t-e^t.2t}{t^4}=\frac{t-2}{t^3}e^t\cos\left(\frac{e^t}{t^2}\right) \text{ as before}$$

Example: If x increases at the rate of $2\ cm/\sec$ at the instant when $x=3\ cm$. and .., at what rate must y be changing in order that the function $2xy-3x^2y$ shall be neither increasing nor decreasing?

Solution: Let $u=2xy-3x^2y$, so that

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}=(2y-6xy)\frac{dx}{dt}+(2x-3x^2)\frac{dy}{dt} \qquad \text{(i)}$$

when $x=3$ and $y=1, \frac{dx}{dt}=2$, and u is neither increasing nor decreasing ,i.e., $\frac{du}{dt}=0$

$\therefore$ (i) becomes $0=(2-6\times 3)2+(2\times 3-3\times 9)\frac{dy}{dt}$

or $\frac{dy}{dt}=-\frac{32}{21}\,cm/\sec$. Thus y is decresing at the rate of $\frac{32}{21}\,cm/\sec$.

Example: If $u=F(x-y,y-z,z-x)$ prove that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Solution: Put $x-y=r, y-z=s$ and $z-x=t$, so that $u=f(r,s,t)$.

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}.(1)+\frac{\partial x}{\partial s}.(0)+\frac{\partial u}{\partial t}.(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t} \qquad \text{(i)}$$

Similarly, $\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$ (ii0

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \qquad \text{(iii)}$$

Adding (i), (ii) and (iii), we get the required result.

4.4 Jacobian

If u and v are two function of two independent variable x and y then the determinant

$J=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial v}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called the Jacobian of u and v with respect to of x and y which is written as

$\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$ similarly for

: If u v and w are two function of two independent variable x , y and z then the determinant

$J=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ is called the Jacobian of u , v and w with respect to of x , y z which is written as $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J\left(\frac{u,v,w}{x,y,z}\right)$

4.4.1 Properties of Jacobian

1. If $J=\frac{\partial(u,v,w)}{\partial(x,y,z)}$ and $J'=\frac{\partial(x,y,z)}{\partial(u,v,w)}$ then $J.J'=1$

2. Chain rule for Jacobian if u,v are function of r,s are function of x,y then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}.\frac{\partial(r,s)}{\partial(x,y)}$$

3. If u_1,u_2,u_3 instead of being given explicitly in terms x_1,x_2,x_3 be connected with them equations such as

$f_1(u_1,u_2,u_3,x_1,x_2,x_3)=0$ $f_2(u_1,u_2,u_3,x_1,x_2,x_3)=0$ $f_3(u_1,u_2,u_3,x_1,x_2,x_3)=0$ then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}=(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}\Big/\frac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}$$

($(-1)^3$ is for three variable system)

4. If u_1,u_2,u_3 be functions of x_1,x_2,x_3 then the necessary and sufficient condition for existence of a functional relationship of the form $f_1(u_1,u_2,u_3)=0$ is $J\left[\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}\right]=0$

Example: In a polar coordinates $x = r\cos\theta$ $y = r\cos\theta$ then find $J\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)$,

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \Rightarrow \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Example: $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$ find the $J\left[\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}\right]$

Solution: $$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1x_3}{x_2^2} & \frac{x_1}{x_1} \\ \frac{x_3}{x_2} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix} =$$

$$\frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_3 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} = -\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

Example: $u = x^2 - y^2, v = 2xy$ $x = r\cos\theta, y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Solution: Use the formula $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}\Rightarrow\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(r,\theta)}=(x^2+y^2)\times r=4r^3 \text{ (where } x=r\cos\theta, y=r\sin\theta\text{)}$$

Example: $u=xyz, v=x^2+y^2+z^2, w=x+y+z$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Solution: We are using property three

$$f_1=u-xyz,\quad f_2=v-x^2-y^2-z^2,\quad f_3=w-x-y-z$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,z)}/\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=J=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z}\end{vmatrix}=\begin{vmatrix}-yz & -xz & -xy\\ -2x & -2y & -2z\\ -1 & -1 & -1\end{vmatrix}\Rightarrow -2(x-y)(y-z)(z-x)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=(-1)^3.1/2(x-y)(y-z)(z-x)$$

4.5 Taylor's Series and Maclaurine Series Expansion

If a function f is such that f^{n-1} is continuous in $[a,a+h]$ and derivative in $]a,a+h[$ then their exist a number $\theta\in]0,1[$ such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a).....+\frac{h^{n-1}}{\lfloor n}f^{n-1}(a)+\frac{h^n}{\lfloor n-1}(1-\theta)^{n-1}f^n(a+\theta h)$$

One can also write Taylor expansion by putting $a+h=x$

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a).....+\frac{(x-a)^{n-1}}{\lfloor n}f^{n-1}(a)$$

$$+\frac{(x-a)^n}{\lfloor n-1}f^{n-1}(a)+f^n(\xi)\frac{(x-a)^n}{\lfloor n}$$

where $f^n(\xi)$ is identified as reminder term $\xi=a+\theta(x-a)$ and $0<\theta<1$

The $(n+1)th$ term of the expansion is $\frac{h^n}{\lfloor n-1}(1-\theta)^{n-1}f^n(a+\theta h)$

4.5.1 Maclaurine's Development

Changing a to 0 and h to x one will get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0).....+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+f^n(\xi)\frac{x^n}{\lfloor n}$$

Where $f^n(\xi)$ is identified as reminder term $\xi=\theta x$ and $0<\theta<1$

Example: In the Taylor series expansion of $\exp x+\sin(x)$ about $x=\pi$ then what is coefficient of $(x-\pi)^2$.

Solution: Taylor series about $x=\pi$

$$f(x)=f(\pi)+(x-\pi)f'(\pi)+\frac{(x-\pi)^2}{\lfloor 2}f''(\pi).....+\frac{(x-\pi)^{n-1}}{\lfloor n}f^{n-1}(\pi)+\frac{(x-\pi)^n}{\lfloor n-1}f^{n-1}(\pi)$$

$$f(x)=\exp x+\sin(x)$$

$$f(\pi)=\exp\pi+\sin(\pi)=\exp\pi$$

$$f'(x)=\exp x+\cos x\Rightarrow f'(\pi)=\exp\pi+\cos\pi=\exp\pi-1$$

$$f''(x)=\exp x-\sin x\Rightarrow f''(\pi)=\exp\pi-\sin\pi\Rightarrow f''(\pi)=\exp\pi$$

The coefficient of $(x-\pi)^2$ is $\frac{1}{2}f''(\pi)=\frac{\exp\pi}{2}$

Example: Expand Maclaurin's series expand of $\tan x$ up to x^3

Solution: $f(x)=\tan x \Rightarrow f(0)=0$

$f'(x)=\sec^2 x \Rightarrow f'(0)=1$

$f''(x)=2\tan x\sec^2 x \Rightarrow f''(0)=0$

$f'''(x)=2\sec^2 x+6\tan^2 x\sec^2 x \Rightarrow f'''(0)=2$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\frac{x^3}{\lfloor 3}f'''(0).....+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+f^n(\xi)\frac{x^n}{\lfloor n}$$

$$\tan x=0+x.1+0.\frac{x^2}{\lfloor 2}+2\frac{x^3}{\lfloor 3}\Rightarrow x+\frac{x^3}{3}$$

Some Important Maclaurin's Expansions

1. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....$ 2. $\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+.....$
3. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-.....$ 4. $\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+.....$
5. $\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+.....$
6. $e^x=1+x+\frac{x^2}{2!}+\frac{x3}{3!}+.....$ 7. $e^{-x}=1-x+\frac{x^2}{2!}-\frac{x3}{3!}+.....$
8. $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}....., \quad \text{if } |x|<1$
9. $\log_e(1-x)=-\left\{x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}....\right\}$

Example: Expand $\exp(\sin x)$ by maclaurin series upto the term containing x^4

Solution: $\exp\sin x=1+\frac{(\sin x)^2}{\lfloor 2}+\frac{(\sin x)^2}{\lfloor 3}+\frac{(\sin x)^2}{\lfloor 4}$

$$=1+\left(x-\frac{x^3}{\lfloor 3}+....\right)+\frac{1}{\lfloor 2}\left(x-\frac{x^3}{\lfloor 3}+....\right)^2+\frac{1}{\lfloor 3}\left(x-\frac{x^3}{\lfloor 3}+....\right)^3.....$$

$$=1+\left(x-\frac{x^3}{6}+....\right)+\frac{1}{2}\left(x^2-\frac{x^3}{\lfloor 3}+....\right)+\frac{1}{6}\left(x^3-...\right)+\frac{1}{24}x^4=1+x+\frac{x^2}{2}-\frac{x^4}{8}$$

Taylor 's theorem for function in two variable .

$$f(x+h,y+k)=f(x,y)+h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+\frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f.....$$

Put $h=x-a\ ,k=y-b$

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{\lfloor 2}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+....$$

$$f(x,y)=f(0,0)+\left[xf_x(0,0)+yf_y(0,0)\right]+\frac{1}{\lfloor 2}\left[x^2 f_{xx}(0,0)+2xybf_{xy}(0,0)+y^2 f_{yy}(0,0)\right]+....$$

Example: Expand $e^x \log(1+y)$ upto powers of x and y upto term of two degree.

Solution: $f(x,y)=e^x\log(1+y)\ \Rightarrow f(0,0)=0$

$$f_x(x,y)=e^x\log(1+y)\Rightarrow f_x(0,0)=0,\quad f_y(x,y)=\frac{e^x}{1+y}\ \Rightarrow f_y(0,0)=1$$

$$f_{xx}(x,y)=e^x\log(1+y)\Rightarrow f_{xx}(0,0)=0,\quad f_{xy}(x,y)=\frac{e^x}{1+y}\ \Rightarrow f_{xy}(0,0)=1$$

$$f_{yy}(x,y)=-\frac{e^x}{(1+y)^2}\ \Rightarrow f_{yy}(0,0)=-1$$

$$e^x\log(1+y)=y+xy-\frac{1}{2}y^2+\frac{1}{2}(x^2y-xy^2).....$$

MCQ (Multiple Choice Questions)

Q1. $f(x)=(1+3x)^{1/3}$ when $\neq 0$ $f(0)=e^3$ is continuous for $x=0$

(a) 0 (b) 1 (c) e^3 (d) e^{-3}

Q2. The limit $\lim_{x\to 0}\frac{1}{\sin^2 x}\int_{x/2}^{x}\sin^{-1}t\,dt$ is equals to

(a) 0 (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{3}{8}$

Q3. If $Y=\left\{\frac{x}{1+|x|}\,x\in R\right\}$, then the set of all limit points of Y is

(a) $(-1,1)$ (b) $(-1,1]$ (c) $[0,1]$ (d) $[1,1]$

Q4. For $m,n\in N$, define $f_{m,n}(x)=\begin{cases}x^m\sin\left(\frac{1}{x^n}\right), & x\neq 0\\ 0, & x=0\end{cases}$

Then, at $x=0$, $f_{m,n}$ is

(a) differentiable for each pair m, n with $m>n$

(b) differentiable for each pair m, n with $m<n$

(c) not differentiable for each pair m, n with $m>n$

(d) not differentiable for each pair m, n with $m<n$

Q5. Let a be a non-zero real number. Then $\lim_{x\to a}\frac{1}{x^2-a^2}\int_a^x\sin\left(t^2\right)dt$ equals

(a) $\frac{1}{2a}\sin\left(a^2\right)$ (b) $\frac{1}{2a}\cos\left(a^2\right)$

(c) $-\frac{1}{2a}\sin\left(a^2\right)$ (d) $-\frac{1}{2a}\cos\left(a^2\right)$

Q6. Let $f:R\to R$ be a continuous function. If $\int_0^x f(2t)dt=\frac{x}{\pi}\sin(\pi x)$ for all $x\in R$, then $f(2)$ is equal to

(a) -1 (b) 0 (c) 1 (d)

Q7. Let $f(x)=(x-2)^{17}(x+5)^{24}$. Then

(a) f does not have a critical point at 2

(b) f has a minimum at 2

(c) f has a maximum at 2

(d) f has neither a minimum nor a maximum at 2

Q8. If $f(x,y)$ is defined by

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^4} & if\ (x,y)\neq(0,0)\\ 0 & if\ (x,y)=(0,0)\end{cases}$$

then

(a) $f_x(0,0)=0$ and $f_y(0,0)=0$ (b) $f_x(0,0)=1$ and $f_y(0,0)=0$

(c) $f_x(0,0)=0$ and $f_y(0,0)=1$ (d) $f_x(0,0)=1$ and $f_y(0,0)=1$

Q9. Which of the following statements holds regarding the continuity and the existence of partial derivatives of f at $(0,0)$?

(a) Both partial derivatives of f exist at $(0,0)$ and f is continuous at $(0,0)$

(b) Both partial derivatives of f exist at $(0,0)$ and f is NOT continuous at $(0,0)$

(c) One partial derivatives of f does NOT exist at $(0,0)$ and f is continuous at $(0,0)$

(d) One partial derivatives of f does NOT exist at $(0,0)$ and f is NOT continuous at $(0,0)$

Q10. Let $f=y^x$ then the value of $\dfrac{\partial^2 f}{\partial x\partial y}$ at $x=2, y=1$ is

(a) 0 (b) $\ln 2$ (c) 1 (d) $\dfrac{1}{\ln 2}$

Q11. If $z = xy\ln(xy)$ then which of the following is correct

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ (b) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

(c) $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$ (d) $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

Q12. Let $f(x,y) = \sum_{K=1}^{10}\left(x^2 - y^2\right)^K$ for all (x,y). Then for all (x,y)

(a) $x\frac{\partial f}{\partial x}(x,y) - y\frac{\partial f}{\partial y}(x,y) = 0$ (b) $x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = 0$

(c) $y\frac{\partial f}{\partial x}(x,y) - x\frac{\partial f}{\partial y}(x,y) = 0$ (d) $y\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) = 0$

Q13. The work done by the force in moving a particle over the circular path $x^2 + y^2 = 1,\ z = 0$ from $(1,0,0)$ to $(0,1,0)$ is

(a) $\pi + 1$ (b) $\pi - 1$ (c) $-\pi + 1$ (d) $-\pi - 1$

Q14. The value of $\int_{z=0}^{1}\int_{y=0}^{z}\int_{x=0}^{y} xy^2z^3\,dxdydz$ is

(a) $\frac{1}{90}$ (b) $\frac{1}{50}$ (c) $\frac{1}{45}$ (d) $\frac{1}{10}$

Q15. The value of $\iint_G \frac{\ln\left(x^2+y^2\right)}{x^2+y^2}dx\,dy$, where $G = \left\{(x,y) \in R^2 : 1 \le x^2 + y^2 \le e^2\right\}$, is

(a) π (b) 2π (c) 3π (d) 4π

Q16. In a cylindrical coordinates $x = r\cos\theta \quad y = r\cos\theta$ and $z = z$

$J\left(\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right)$ is given by

(a) r (b) z (c) $-r$ (d) $-z$

Q17. If $x = u(1-v)$, $y = uv$ then $J\left(\frac{\partial(x,y)}{\partial(u,v)}\right)$ is given by

(a) u (b) v (c) $-u(1-2v)$ (d) $u(1-2v)$

Q18. If $u = x^2 - 2y^2 \quad v = 2x^2 - y^2 \quad x = r\cos\theta, y = r\sin\theta \quad J\left(\frac{\partial(u,v)}{\partial(r,\theta)}\right)$

(a) $6r^3\sin 2\theta$ (b) $12r^3\sin 2\theta$ (c) $6r^3\sin\theta$ (d) $12r^3\sin\theta$

Q19. In the Taylor series expansion of e^x about $x = 2$ the coefficient of $(x-2)^2$

(a) $\frac{1}{\lfloor 4}$ (b) $\frac{2^4}{\lfloor 4}$ (c) $\frac{e^2}{\lfloor 4}$ (d) $\frac{e^4}{\lfloor 4}$

Q20. Which of the following is expansion $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ up to second degree term .

(a) $\frac{\pi}{4} - \frac{1}{2}[(x-1)-(y-1)] + \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right]$

(b) $\frac{\pi}{4} + \frac{1}{2}[(x-1)-(y-1)] + \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right]$

(c) $\frac{\pi}{4} + \frac{1}{2}[(x-1)-(y-1)] - \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right]$

(d) $\frac{\pi}{4} - \frac{1}{2}[(x-1)-(y-1)] - \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right]$

Q21. The Taylor series expansion of $\frac{\sin x}{\pi - x}$ at $x = \pi$

(a) $1 + \frac{(x-\pi)^2}{\lfloor 3} \ldots\ldots$ (b) $-1 - \frac{(x-\pi)^2}{\lfloor 3} \ldots\ldots$

(c) $1 - \frac{(x-\pi)^2}{\lfloor 3} \ldots\ldots$ d) $-1 + \frac{(x-\pi)^2}{\lfloor 3} \ldots\ldots$

Q22. The coefficient of $(x-1)^2$ in the Taylor series expansion of $f(x) = xe^x (x \in R)$ about the point $x = 1$ is

(a) $\frac{e}{2}$ (b) $2e$ (c) $\frac{3e}{2}$ (d) $3e$

NAT (Numerical Answer Type)

Q23. The maximum value of $f(x)=\ln(1+x)-x$ where $(x>-1)$ occur at $x=$ _____

Q24. Let $A(t)$ denote the area bounded by the curve $y=e^{-|x|}$, the x-axis and the straight lines $x=-t$ and $x=t$. Then $\lim_{t\to\infty}A(t)$ is equal to _____

Q25. Value of the countour on the plane ,where the partial derivative of x^2+y^2 with respect to y is equal to the partial derivative of $6y+4x$ with respect to x is $y=$ _____

Q26. If $u=x^3+y^3+3xy^2$ and $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ then value of $n=$_____.

Q27. For all (x,y), let $f(x,y)=\begin{cases}\frac{x}{|x|}\sqrt{x^2+y^2} & if\ x\neq 0\\ 0 & if\ x=0\end{cases}$.

Then $\frac{\partial f}{\partial x}(0,0)+\frac{\partial f}{\partial y}(0,0)$ equals _____.

Q28. Let f has a maximum at $f(x,y)=x^5y^2\tan^{-1}\left(\frac{y}{x}\right)$. Then $x\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=nf$ then value of n is …….

Q29. The value of integration $\int_{x=0}^{1}\int_{y=0}^{x^2}\int_{z=0}^{y}(y+2z)\,dz\,dy\,dx$ is _____.

Q30. If $u=x+3y^2-z^2,\ v=4x^2yz,\ w=2z^2-xy$ then value of $J=\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$ is _____.

Q31. In a spherical coordinates $x=r\sin\theta\cos\phi$ $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$

$J\left(\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right)$ is given by $r^{\alpha}\sin\theta$ then value α is given by _____.

Q32. For the function e^{-x} the linear approximation around $x=2$ is $(3-\alpha x)e^{-2}$ then the value of α is given by _____.

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: $Lt \;\; x \to 0\left[(1+3x)^{1/3x} \right]^3 = e^3$

Ans. 2: (d)

Solution: $\lim_{x\to 0} \frac{1}{\sin^2 x} \int_{x/2}^{x} \sin^{-1} t \, dt = L$

$$\lim_{x\to 0^+} \frac{\int_{x/2}^{x} \sin^{-1} t \, dt}{\sin^2 x} \left[\frac{0}{0} case \right]$$

By L Hospital's rule and using differentiation of integration, we get

$$L = \lim_{x\to 0^+} \frac{\left(\sin^{-1} x - \frac{1}{2} \sin^{-1} \frac{x}{2} \right)/x}{2\left(\frac{\sin x}{x} \right) \cos x} = \frac{1-\frac{1}{4}}{2} = \frac{3}{8}$$

Ans. 3: (d)

Solution: $Y = \left\{ \frac{x}{1+|x|} \, x \in R \right\} \Rightarrow Y(-1,1)$

Because $\lim_{x\to -\infty} \frac{x}{1+|x|} = -1$ and $\lim_{x\to \infty} \frac{x}{1+|x|} = 1$. Hence $Y = Y' = [-1, 1]$

Ans. 4: (a)

Solution: $\underset{x\to 0}{lt} \frac{x^m \sin\left(\frac{1}{x^n} \right) - 0}{x-0} = \underset{x\to 0}{lt} \; x^{m-1} \sin\left(\frac{1}{x^n} \right) = 0$ if $m-1>0$

$\Rightarrow m>1$ if $m>1$ then $f_{m,n}(x)$ is differentiable which holds if $m>n$

Ans. 5: (a)

Solution: $\lim_{x\to a}\frac{\int_a^x \sin\left(t^2\right)dt}{x^2-a^2}\left[\frac{0}{0}case\right]$. By L' Hospital's rule, we get

$\lim_{x\to a}\frac{\sin\left(x^2\right)}{2x}$ (by Leibnitz Theorem)

$=\frac{\sin\left(a^2\right)}{2a}=\frac{1}{2a}\sin\left(a^2\right)$

Ans. 6: (a)

Solution: $\int_0^x f\left(2t\right)dt=\frac{x}{\pi}\sin\left(\pi x\right)$

Differentiating both sides w.r.t. x, we get $f\left(2x\right)=\frac{1}{\pi}\sin\pi x+x\cos\pi x$

Putting $x=1$, we get $f\left(2\right)=-1$

Ans. 7: (d)

Solution: $f\left(x\right)=\left(x-2\right)^{17}\left(x+5\right)^{24}$

$\Rightarrow$ for $x<2$, $f\left(x\right)<0$ and for $x>2$ $f\left(x\right)>0$

so, $x=2$ is neither maxima nor minima point.

Ans. 8: (b)

Solution: $f_x\left(0,0\right)=\underset{h\to 0}{Lt}\frac{f\left(h,0\right)-f\left(0,0\right)}{h}=\underset{h\to 0}{Lt}\frac{h^3}{h^3}=1$

Let $f:R^2\to R$ be defined by

$$f\left(x,y\right)=\begin{cases}\frac{x^2y}{x^4+y^2} & \text{if } \left(x,y\right)\neq\left(0,0\right)\\ 0 & \text{if } \left(x,y\right)=\left(0,0\right)\end{cases}$$

Ans. 9: (b)

Solution: $y = x^2$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$. So, f is not continuous at $(0,0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{x\to 0} \frac{0}{x^4} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

so, both partial derivatives exist.

Ans. 10: (c)

Solution: $f = y^x$, $\frac{\partial f}{\partial y} = xy^{x-1}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(y^{x-1}x\right) = y^{x-1} + xy^{x-1}\ln y$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x=2,y=1} = 1$$

Ans. 11: (c)

Solution: $\frac{\partial z}{\partial x} = y\ln(xy) + \frac{xy}{xy}y \Rightarrow y(\ln xy + 1) \Rightarrow x\frac{\partial z}{\partial x} = xy(\ln(xy)+1)$

$$\frac{\partial z}{\partial y} = x\ln(xy) + \frac{xy}{xy}x \Rightarrow x(\ln xy + 1) \Rightarrow y\frac{\partial z}{\partial y} = xy(\ln(xy)+1)$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

Ans. 12: (d)

Solution: $f(x,y) = \sum_{K=1}^{10}\left(x^2-y^2\right)^K \Rightarrow \frac{\partial f}{\partial x}(x,y) = \sum_{K=1}^{10} K\left(x^2-y^2\right)^{K-1}(2x)$ (i)

and $\frac{\partial f}{\partial y}(x,y) = \sum_{K=1}^{10} K\left(x^2-y^2\right)^{K-1}(-2y)$ (ii)

$$\Rightarrow y\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) = 0$$

Ans. 13: (d)

Solution: $d\vec{r} = dx\hat{i} + dy\hat{j}$

$x^2 + y^2 = 1 \Rightarrow 2xdx + 2ydy = 0$

$\vec{F} \cdot d\vec{r} = 4ydx - 3xydy = 4\sqrt{1-x^2}dx + 3x^2dx$

Work done $w = \int \vec{F}.d\vec{r}$

$$= \int_1^0 \left(4\sqrt{1-x^2} + 3x^2\right)dx = 4\left(\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right) + x^3\Big|_1^0 = -\pi - 1$$

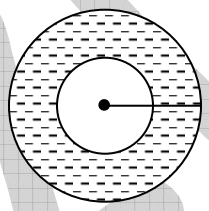
Ans. 14: (a)

Solution: $\int_{z=0}^{1}\int_{z=0}^{z}\int_{x=0}^{y} xy^2z^3 dxdydz = \int_{z=0}^{1}\int_{y=0}^{z}\frac{y^4}{2}z^3 dydz = \int_{z=0}^{1}\frac{z^8}{10} = \frac{1}{90}$

Ans. 15: (b)

Solution: $G = \left\{(x,y) \in R^2 : 1 \le x^2 + y^2 \le e^2\right\}$

$\left\{(r,\theta) : 1 \le r \le e \text{ and } 0 \le \theta \le 2\pi\right\}$



Also $dxdy = rd\theta dr$

$$\iint_G \frac{\ln\left(x^2+y^2\right)}{x^2+y^2}dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=1}^{e}\frac{\ln\left(r^2\right)}{r^2}rd\theta dr = \int_{r=1}^{e}\frac{2\ln r}{r}dr\int_{\theta=0}^{2\pi}d\theta = (\ln r)^2\Big|_1^e.\theta\Big|_0^{2\pi} = 2\pi$$

Ans. 16: (a)

Solution: $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$ and $\frac{\partial z}{\partial r} = 0, \frac{\partial z}{\partial \theta} = 0, \frac{\partial z}{\partial z} = 1,$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Ans. 17: (a)

Solution: $J\left(\frac{\partial(x,y)}{\partial(u,v)}\right)=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=(u-uv)-(-uv)=u$

Ans. 18: (c)

Solution: $\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial v}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & -4y\\ 4x & -2y\end{vmatrix}=-4xy+16xy=12xy \quad \text{put } x=r\cos\theta,\ y=r\sin\theta$$

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}\Rightarrow\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(r,\theta)}=12xy.r=6r^3\sin 2\theta$$

Ans. 19: (c)

Solution: coefficient of $(x-2)^4=\frac{f^4(2)}{\lfloor 4}=\frac{e^2}{\lfloor 4}$

Ans. 20: (a)

Solution: $f(x,y)=\tan^{-1}(y/x) \quad f(1,1)=\tan^{-1}(1)=\frac{\pi}{4}$

$$f_x=\frac{-y}{x^2+y^2}\Rightarrow f_x(1,1)=-\frac{1}{2} \quad , \quad f_y=\frac{x}{x^2+y^2}\Rightarrow f_x(1,1)=\frac{1}{2}$$

$$f_{xx}=\frac{2xy}{(x^2+y^2)^2}\Rightarrow f_{xx}(1,1)=\frac{1}{2} \quad , \quad f_{xy}=\frac{y^2-x^2}{x^2+y^2}\Rightarrow f_{xy}(1,1)=0$$

$$f_{yy}=\frac{-2xy}{(x^2+y^2)^2}\Rightarrow f_{xx}(1,1)=-\frac{1}{2}$$

$$f(x,y) = f(1,1) + \left[(x-1) f_x(1,1) + (y-1) f_y(1,1)\right] +$$

$$\frac{1}{\lfloor 2}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \ldots.$$

$$\frac{\pi}{4} - \frac{1}{2}\left[(x-1)-(y-1)\right] + \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right]$$

Ans. 21: (b)

Solution: $f(x) = f(\pi) + \frac{x-\pi}{\lfloor 1} f'(\pi) + \frac{(x-\pi)^2}{\lfloor 2} f''(\pi)$

$f(\pi) = {}_{Lt(x\to\pi)} \frac{\sin x}{x-\pi} = \frac{0}{0}$ using L' hospitals rule $Lt(x\to\pi)\frac{\cos x}{1} = -1$

$f'(\pi) = 0 \qquad f''(\pi) = -\frac{1}{6} \quad f(x) = -1 - \frac{(x-\pi)^2}{\lfloor 3} \ldots..$

Correct choice is (b)

Ans. 22: (c)

Solution: $f(x) = xe^x = (x-1+1)e^{x-1+1} = \left[1+(x-1)\right]e^{(x-1)} \cdot e$

$$e\left[1+(x-1)\right]\left[1+(x-1)+\frac{(x-1)^2}{2!}+\ldots.\right] = e\left(1+\frac{1}{2!}\right) = \frac{3e}{2}$$

Hence, coefficient of $(x-1)^2$ will be $e\left(1+\frac{1}{2!}\right) = \frac{3e}{2}$

NAT (Numerical Answer Type)

Ans. 23: 0

Solution: $f'(x) = \frac{1}{1+x} - 1 = 0$

$$\frac{1-1-x}{1+x} = 0 \Rightarrow \frac{x}{1+x} = 0 \Rightarrow x = 0 \quad f''(0) = -1 < 0$$

Ans. 24: 2

Solution: $A(t)=\int_{-t}^{t} y\,dx=\int_{-t}^{t} e^{-|x|}dx$

$=2\int_{0}^{t} e^{-x}dx$; as $e^{-|x|}$ is even function.

$=2\left(-e^{-x}\right)\Big|_{0}^{t}=2\left(1-e^{-t}\right)$

$\lim_{t\to\infty} A(t)=\lim_{t\to\infty} 2\left(1-e^{-t}\right)=2$

Ans. 25: 2

Solution: $\frac{\partial(x^2+y^2)}{\partial y}=2y$ and $\frac{\partial(6y+4x)}{\partial x}=4$ $2y=4\Rightarrow y=2$

Ans. 26: 3

Solution: $u=x^3+y^3+3xy^2$ is homogeneous function of degree 3 so $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$

Ans. 27: 1

Solution: $\frac{\partial f}{\partial x}(0,0)=\underset{h\to 0}{lt}\frac{\frac{h}{|h|}\sqrt{h^2}}{h}=1$

$\frac{\partial f}{\partial y}(0,0)=\lim_{k\to 0}\frac{0-0}{k}=0$

$\frac{\partial f}{\partial x}(0,0)+\frac{\partial f}{\partial y}(0,0)=1+0=1$

Ans. 28: 7

Solution: $f(x,y)=x^5y^2\tan^{-1}(y/x)$

$\Rightarrow \frac{\partial f}{\partial x}=5x^4y^2\tan^{-1}(y/x)+x^5y^2\frac{1}{1+\left(\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right)$

$\Rightarrow \frac{\partial f}{\partial x}=5x^4y^2\tan^{-1}(y/x)-\frac{x^5y^3}{x^2+y^2}$ (i)

$$\frac{\partial f}{\partial y} = 2x^5 y \tan^{-1}\frac{y}{x} + x^5 y^2 \frac{1}{\left(1+\frac{y^2}{x^2}\right)} \times \frac{1}{x}$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2x^5 y \tan^{-1}\frac{y}{x} + \frac{x^6 y^2}{x^2+y^2} \qquad \text{(ii)}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 7x^5 y^2 \tan^{-1}(y/x) = 7f$$

Ans. 29: 0.095

Solution: $\int_{x=0}^{1}\int_{y=0}^{x^2}\int_{z=0}^{y}(y+2z)dzdydx = \int_{x=0}^{1}\int_{y=0}^{x^2} yz+z^2\Big|_0^y dydx$

$$= \int_{x=0}^{1}\int_{y=0}^{x^2} 2y^2 dy\,dx = \int_{x=0}^{1}\frac{2y^3}{3}\Big|_0^{x^2} = \int_{x=0}^{1}\frac{2x^6}{3}dx = \frac{2x^7}{21}\Big|_0^1 = \frac{2}{21}$$

Ans. 30: 20

Solution: $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xy & 4x^2 z & 4x^2 y \\ -y & -x & 4z \end{vmatrix}$ at $(1,-1,0)$ $J = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 20$

Ans. 31: 2

Solution: $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$, $\frac{\partial z}{\partial r} = \cos\theta$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \quad \text{and} \quad , \frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial \phi} = -r\cos\theta\sin\phi \qquad \frac{\partial y}{\partial \phi} = r\cos\theta\cos\phi \quad , \frac{\partial z}{\partial \phi} = 0$$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix}=\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\cos\theta\sin\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}=r^2\sin\theta$$

Ans. 32: 1

Solution: The linear approximation $f(2)+(x-2)f'(2)=(3-x)e^{-2}$ so $\alpha=1$

fiziks

5. Order and Degree of a Differential Equations

The order of a differential equation is the order of the highest differential co-efficient present in the equation.

Example: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=0$ is a second order differential equation.

The degree of a differential equation is the degree of the highest derivative after removing the radical sign and fraction.

Example: $\left(\frac{d^2y}{dx^2}\right)^2+2\left(\frac{dy}{dx}\right)^3+3y=0$ has degree of 2.

5.1 Differential Equations of the First Order and First Degree

5.1.1 Separation of the variables

If a differential equation can be written in the form

$$f(y)dy=\phi(x)dx$$

We say that variables are separable, y on the left hand side and x on the right hand side.

We get the solution by integrating both sides.

Example: Solve the differential equation $(yx^2+y)dy+(xy^2+x)dx=0$.

Solution: $(yx^2+y)\frac{dy}{dx}+(xy^2+x)=0\Rightarrow\frac{y\,dy}{y^2+1}=-\frac{x}{x^2+1}dx$

$$\Rightarrow\frac{1}{2}\log(y^2+1)=-\frac{1}{2}\log(x^2+1)+c\Rightarrow(y^2+1)(x^2+1)=c$$

Example: Solve the differential equation $x^4\frac{dy}{dx}+x^3y=-\sec(xy)$.

Solution: $x^4\frac{dy}{dx}+x^3y=-\sec(xy)\Rightarrow x^3\left(x\frac{dy}{dx}+y\right)=-\sec xy$

Put $v=xy\Rightarrow\frac{dv}{dx}=x\frac{dy}{dx}+y\Rightarrow x^3\frac{dv}{dx}=-\sec v\Rightarrow\frac{dv}{\sec v}=-\frac{dx}{x^3}$

$$\Rightarrow\int\cos v\,dv=-\int\frac{dx}{x^3}+c\Rightarrow\sin v=\frac{1}{2x^2}+c\Rightarrow\sin xy=\frac{1}{2x^2}+c$$

5.1.2 Homogeneous Equation

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$$

is called a homogeneous equation if each term of $f(x,y)$ and $\phi(x,y)$ is of the same degree.

Example: Find the solution of the differential equation $(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy$.

Solution: $(2xy + x^2)\frac{dy}{dx} = 3y^2 + 2xy \Rightarrow \frac{dy}{dx} = \frac{3y^2 + 2xy}{2xy + x^2}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = \frac{3v^2x^2 + 2vx^2}{2vx^2 + x^2} = \frac{3v^2 + 2v}{2v + 1}$

$$\Rightarrow x\frac{dv}{dx} = \frac{3v^2 + 2v}{2v+1} - v = \frac{v^2 + v}{2v+1} \Rightarrow \left(\frac{2v+1}{v^2+v}\right)dv = \frac{1}{x}dx \Rightarrow \log(v^2 + v) = \log x + \log c$$

$$\Rightarrow v^2 + v = cx \Rightarrow y^2 + xy = cx^3.$$

5.1.3 Equations Reducible to homogeneous form

The equations of the form $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$ can be reduced to homogeneous form by the substitutions $x = X + h, y = Y + k$ (h, k being constants)

$$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + C}$$

Choose h, k so that $\left.\begin{matrix} ah + bk + c = 0 \\ Ah + Bk + C = 0 \end{matrix}\right\} \Rightarrow \frac{dy}{dx} = \frac{aX + bY}{AX + BY}$

Case of failure: $\frac{a}{A} = \frac{b}{B} = \frac{1}{m} \Rightarrow \frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + C}$

Now put $ax + by = z$ and apply the method of separation of variables.

Example: Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Solution: Put $x=X+h, y=Y+k$ (h,k being constants)

The given equation reduces to $\frac{dY}{dX}=\frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$

Now choose h,k so that $\left.\begin{matrix}h+2k-3=0\\2h+k-3=0\end{matrix}\right\}\Rightarrow h=k=1.$

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

Put $Y=vX$ so that $\frac{dY}{dX}=v+X\frac{dv}{dX}\Rightarrow v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}=\frac{1+2v}{2+v}$

$$\Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1-v^2}{2+v}\Rightarrow\left(\frac{2+v}{1-v^2}\right)dv=\frac{dX}{X}\Rightarrow\frac{1}{2}\frac{1}{(1+v)}dv+\frac{3}{2}\frac{1}{1-v}dv=\frac{dX}{X}$$

On integrating we have

$$\frac{1}{2}\log(1+v)-3/2\log(1-v)=\log X+\log C\Rightarrow\log\frac{1+v}{(1-v)^3}=\log C^2X^2\Rightarrow\frac{1+v}{(1-v)^3}=C^2X^2$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3}=C^2X^2\Rightarrow\frac{X+Y}{(X-Y)^3}=C^2\Rightarrow X+Y=C^2(X-Y)^3$$

Put $X=x-1, Y=y-1\Rightarrow x+y-2=a(x-y)^3.$

Example: Solve $\frac{dy}{dx}=\frac{x+2y-1}{x+2y+1}$

Solution: Put $x+2y=z\Rightarrow 1+2\frac{dy}{dx}=\frac{dz}{dx}\Rightarrow\frac{1}{2}\frac{dz}{dx}-\frac{1}{2}=\frac{z-1}{z+1}\Rightarrow\frac{z+1}{3z-1}dz=dx$

$$\Rightarrow\left(\frac{1}{3}+\frac{4}{3}\frac{1}{3z-1}\right)dz=dx\Rightarrow\frac{z}{3}+\frac{4}{9}\log(3z-1)=x+C\Rightarrow 3z+4\log(3z-1)=9x+9C$$

$$\Rightarrow 3(x+2y)+4\log(3x+6y-1)=9x+9C\Rightarrow 3x-3y+a=2\log(3x+6y-1)$$

5.1.4 Linear Differential Equations

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \text{ where } P \text{ and } Q \text{ are function of } x \text{ (but not } y\text{) or constant}$$

$$\Rightarrow y \times I.F. = \int (Q \times I.F.)dx + c \text{ where } I.F. = e^{\int P dx}.$$

Example: Solve $x(x-1)\frac{dy}{dx} - (x-2)y = x^2(2x-1)$

Solution:

$$\frac{dy}{dx} - \frac{(x-2)}{x(x-1)}y = \frac{x^2(2x-1)}{x(x-1)}$$

$$\Rightarrow I.F. = e^{\int -\frac{(x-2)}{x(x-1)}dx} = e^{\int \left[\frac{1}{(x-1)} - \frac{2}{x}\right]dx} = e^{\log(x-1) - 2\log x} = e^{\log\frac{(x-1)}{x^2}} = \frac{(x-1)}{x^2}$$

$$\Rightarrow y \times \frac{(x-1)}{x^2} = \int \left(\frac{x^2(2x-1)}{x(x-1)} \times \frac{(x-1)}{x^2} \right) dx + c = \int \frac{(2x-1)}{x} dx + c = 2x - \log x + c$$

Example: Solve $\frac{dy}{dx} + 2y\tan x = \sin x$ when $y = 0$ for $x = \frac{\pi}{3}$.

Solution: $\frac{dy}{dx} + 2y\tan x = \sin x$

$$I.F. = e^{2\int \tan x\, dx} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x$$

Hence, solution is

$$y \cdot \sec^2 x = \int \sec^2 x \cdot \sin x\, dx + c = \int \tan x \sec x\, dx + c$$

$$\Rightarrow y\sec^2 x = \sec x + c$$

Put $x = \frac{\pi}{3}, y = 0 \Rightarrow 0 \times \sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + c \Rightarrow c = -2 \Rightarrow y\sec^2 x = \sec x - 2$

$$\Rightarrow y = \cos x - 2\cos^2 x$$

5.1.5 Equation Reducible to Linear Form

A differential equation of the form $\frac{dy}{dx}+Py=Qy^n$

where P and Q are function of x (but not y) or constant can be reduced to the linear form on dividing by y^n and substituting $\frac{1}{y^{n-1}}=z$.

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx}+P\frac{1}{y^{n-1}}=Q$$

$$\because \frac{1}{y^{n-1}}=z \Rightarrow \frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{1}{1-n}\frac{dz}{dx}+Pz=Q$$

$\Rightarrow \frac{dz}{dx}+P(1-n)z=Q(1-n)$ which is a linear differential equation.

Example: $x\frac{dy}{dx}+y\log y=xye^x$

Solution: Dividing by xy, we get $\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x$.

Put $\log y=z \Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x \Rightarrow \frac{dz}{dx}+\frac{z}{x}=e^x$

$$I.F.=e^{\int\frac{1}{x}dx}=e^{\log x}=x \Rightarrow zx=xe^x-e^x+c \Rightarrow x\log y=xe^x-e^x+c.$$

Example: Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$

Solution: $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y \Rightarrow \cos y\frac{dy}{dx}-\frac{\sin y}{1+x}=(1+x)e^x$

Put $\sin y=z$, so that $\cos y\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dz}{dx}-\frac{z}{1+x}=(1+x)e^x$

$$I.F.=e^{-\int\frac{1}{1+x}dx}=e^{-\log(1+x)}=e^{\log\frac{1}{1+x}}=\frac{1}{1+x}$$

$$\Rightarrow z\frac{1}{1+x}=\int(1+x)e^x\cdot\frac{1}{1+x}dx+c=\int e^x dx+c \Rightarrow \frac{\sin y}{1+x}=e^x+c$$

5.1.6 Exact Differential Equation

A differential equation of the form $Mdx + Ndy = 0$ is said to be an exact differential equation if it satisfies the following condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$\int_{y=\text{constant}} Mdx + \int (\text{terms of N not containing } x)dy = C$$

Example: Solve $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

Solution: $M = \left(y^2 e^{xy^2} + 4x^3\right),\ N = \left(2xye^{xy^2} - 3y^2\right)$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2xy^3.e^{xy^2} + 2y.e^{xy^2}$$

$$\Rightarrow \left(e^{xy^2} + x^4\right) + \left(-y^3\right) = C$$

5.1.7 Equations Reducible to the Exact Form

Sometimes a differential equation which is not exact may become so, on multiplication by a suitable function known as the integrating factor.

(a) If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$ then $\text{I.F.} = e^{\int f(x)dx}$.

Example: solve $(2x\log x - xy)dy + 2ydx = 0$

Solution: $M = 2y,\ N = (2x\log x - xy)$

$$\frac{\partial M}{\partial y} = 2,\ \frac{\partial N}{\partial x} = 2(1+\log x) - y \Rightarrow \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = -\frac{1}{x} = f(x) \text{ then } \text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Multiplying by $\dfrac{1}{x}$ we get $\dfrac{2y}{x}dx + (2\log x - y)dy = 0$ which is an exact equation.

$$\Rightarrow \int \frac{2y}{x}dx + \int (-y)dy = C \Rightarrow 2y\log x - \frac{y^2}{2} = C$$

(b) If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a function of y alone, say $f(y)$ then $\text{I.}F.=e^{\int f(y)dy}$.

Example: solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

Solution: $M=\left(y^4+2y\right),\ N=\left(xy^3+2y^4-4x\right)\Rightarrow\dfrac{\partial M}{\partial y}=4y^3+2,\ \dfrac{\partial N}{\partial x}=y^3-4$

$$\Rightarrow\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=-\frac{3}{y}=f(y)\text{ then }\text{I.}F.=e^{\int-\frac{3}{y}dy}=e^{-3\log y}=\frac{1}{y^3}$$

Multiplying by $\dfrac{1}{y^3}$ we get $\dfrac{1}{y^3}\left(y^4+2y\right)dx+\dfrac{1}{y^3}\left(xy^3+2y^4-4x\right)dy=0$ which is an exact equation.

$$\Rightarrow\int\left(y+\frac{2}{y^2}\right)dx+\int(2y)dy=C\Rightarrow x\left(y+\frac{2}{y^2}\right)+y^2=C$$

(c) If $M=yf_1(xy)$ and $N=xf_2(xy)$ then $\text{I.}F.=\dfrac{1}{Mx-Ny}$.

Example: solve $y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$

Solution:

Divide by xy, we get $y(1+2xy)dx+x(1-xy)dy=0\Rightarrow M=yf_1(xy)$ and $N=xf_2(xy)$

$$\text{I.}F.=\frac{1}{Mx-Ny}=\frac{1}{xy(1+2xy)-xy(1-xy)}=\frac{1}{3x^2y^2}.$$

Multiplying $y(1+2xy)dx+x(1-xy)dy=0$ by $\dfrac{1}{3x^2y^2}$ we get an exact equation.

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0$$

$$\Rightarrow\int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int\left(-\frac{3}{y}\right)dy=C\Rightarrow-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log=C$$

(d) If $Mdx + Ndy = 0$ be a homogeneous equation in x and y then $\text{I.F.} = \frac{1}{Mx + Ny}$.

Example: solve $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$

Solution: $\left(x^3 + y^3\right)dx - xy^2 dy = 0$

Here $M = x^3 + y^3$, $N = -xy^2$, $\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{\left(x^3 + y^3\right)x + \left(-xy^2\right)y} = \frac{1}{x^4}$

Multiplying by $\frac{1}{x^4}$ we get $\frac{1}{x^4}\left(x^3 + y^3\right)dx - \frac{xy^2}{x^4}dy = 0$

or $\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$ which is an exact equation.

$$\int\left(\frac{1}{x} + \frac{y^2}{x^4}\right)dx = C \Rightarrow \log x - \frac{y^3}{3x^3} = C$$

5.2 Linear Differential Equations of Second Order with Constant Coefficients

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \text{ or } \left(D^2 + PD + Q\right)y = R$$

where P, Q and R are function of x or constant.

Its complete solution = Complementary Function+ Particular Integral

Methods for finding Complementary Function

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ or $\left(D^2 + PD + Q\right) = 0$

(a) Roots $\left(m_1, m_2\right)$, Real and Different.

Then $C.F. = C_1e^{m_1x} + C_2e^{m_2x}$

Example: Find complementary function for $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$.

Solution: $\left(D^2 - 8D + 15\right) = 0 \Rightarrow D = 3,5 \Rightarrow C.F. = C_1e^{3x} + C_2e^{5x}$

(b) Roots (m_1, m_2), Real and Equal.

Then $C.F. = (C_1 + C_2 x)e^{m_1 x}$

Example: Find complementary function for $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$.

Solution: $(4D^2 - 4D + 1) = 0 \Rightarrow D = \frac{1}{2}, \frac{1}{2} \Rightarrow C.F. = (C_1 + C_2 x)e^{\frac{1}{2}x}$

(c) Roots (m_1, m_2) Imaginary

Then $C.F. = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

Example: Find complementary function for $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$.

Solution: $(D^2 + 4D + 5) = 0 \Rightarrow D = -2 + i,\ -2 - i \Rightarrow C.F. = e^{-2x}[A\cos x + B\sin x]$

Methods for finding Particular Integral

(a) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$. If $f(a) = 0$ then $\frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{f'(a)} \cdot e^{ax}$

If $f'(a) = 0$ then $\frac{1}{f(D)}e^{ax} = x^2 \cdot \frac{1}{f''(a)} \cdot e^{ax}$.

Example: Find particular integral for $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$.

Solution: $P.I. = \frac{1}{D^2 + 6D + 9}.5e^{3x} = \frac{5e^{3x}}{3^2 + 6.3 + 9} = \frac{5e^{3x}}{36}$

Example: Find particular integral for $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 7e^{-2x} - \log 2$.

Solution: $\Rightarrow P.I. = \frac{1}{D^2 - 6D + 9}.7e^{-2x} + \frac{1}{D^2 - 6D + 9}.(-\log 2)$

$\Rightarrow P.I. = \frac{7e^{-2x}}{4 + 12 + 9} - \frac{\log 2 \times e^{0x}}{0 - 0 + 9} = \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$

Example: Find particular integral for $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=e^{3x}$.

Solution: $\because\ f(D)=D^2-6D+9\ \Rightarrow f(3)=9-18+9=0$, $f'(3)=6-6=0$

$$\Rightarrow P.I.=\frac{1}{D^2-6D+9}.e^{3x}=x^2\frac{1}{2}e^{3x}$$

(b) $\frac{1}{f(D)}x^n=[f(D)]^{-1}x^n$. Expand $[f(D)]^{-1}$ and then operate.

Example: Find particular integral for $(D^2+5D+4)y=3-2x$.

Solution: $P.I.=\left[D^2+5D+4\right]^{-1}(3-2x)=\frac{1}{4}\left[1+\frac{5}{4}D+\frac{5}{4}D^2\right]^{-1}(3-2x)$

$$\Rightarrow P.I.=\frac{1}{4}\left[1-\frac{5}{4}D-\frac{5}{4}D^2\right](3-2x)=\frac{1}{4}\left[3-2x-\frac{5}{4}\times-2\right]=\frac{1}{8}[11-4x]$$

(c) $\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax$ and $\frac{1}{f(D^2)}\cos ax=\frac{1}{f(-a^2)}\cos ax$

If $f(-a^2)=0$ then $\frac{1}{f(D^2)}\sin ax=x.\frac{1}{f'(-a^2)}\sin ax$.

Example: Find particular integral for $(D^2+4)y=\sin 3x$.

Solution: $P.I.=\frac{1}{D^2+4}.\sin 3x=\frac{\sin 3x}{(-3^2)+4}=-\frac{1}{5}\sin 3x$

Example: Find particular integral for $(D^2+4)y=\cos 2x$.

Solution: $\because\ f(D^2)=(D^2+4)\Rightarrow f(-2^2)=(-4+4)=0,\ f'(-2^2)\neq 0$

$$\Rightarrow P.I.=\frac{1}{D^2+4}.\cos 2x=x\frac{1}{2D}.\cos 2x=\frac{x}{2}\left(\frac{1}{2}\sin 2x\right)=\frac{x}{4}\sin 2x.$$

Example: Find particular integral for $(D^2+D+1)y=\cos 2x$.

Solution: $P.I.=\frac{1}{D^2+D+1}.\cos 2x=\frac{1}{-2^2+D+1}.\cos 2x=\frac{1}{D-3}.\cos 2x$.

$$\Rightarrow P.I.=\frac{D+3}{D^2-9}.\cos 2x=\frac{D+3}{-2^2-9}.\cos 2x=\frac{1}{13}(2\sin 2x-3\cos 2x)$$

(d) $\frac{1}{f(D^2)}e^{ax}\phi(x)=e^{ax}\frac{1}{f(D+a)}\phi(x)$

Example: Find particular integral for $(D^2-4D+4)y=x^3e^{2x}$.

Solution: $P.I.=\frac{1}{D^2-4D+4}.x^3e^{2x}=e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}.x^3$.

$\Rightarrow P.I.=e^{2x}\frac{1}{D^2}.x^3=e^{2x}\frac{1}{D}\left(\frac{x^4}{4}\right)=e^{2x}\frac{x^5}{20}$

Example: Find particular integral for $(D^2-5D+6)y=e^x\cos 2x$.

Solution: $P.I.=\frac{1}{D^2-5D+6}e^x\cos 2x=e^x\frac{1}{(D+1)^2-5(D+1)+6}\cos 2x$.

$=e^x\frac{1}{D^2-3D+2}\cos 2x\Rightarrow P.I.=e^x\frac{1}{-4-3D+2}\cos 2x=-e^x\frac{1}{3D+2}\cos 2x$

$=-e^x\frac{3D-2}{9D^2-4}\cos 2x=-e^x\frac{3D-2}{-36-4}\cos 2x$

$\Rightarrow P.I.=\frac{e^x}{40}(3D-2)\cos 2x=\frac{e^x}{40}(-6\sin 2x-2\cos 2x)=-\frac{e^x}{20}(3\sin 2x+\cos 2x)$

Example: Find particular integral for $(D^2-6D+13)y=2^x$.

Solution: $\Rightarrow P.I.=\frac{1}{D^2-6D+13}2^x=\frac{1}{D^2-6D+13}e^{x\log 2}$

$=e^{x\log 2}\frac{1}{(\log 2)^2-6(\log 2)+13}\Rightarrow P.I.=\frac{2^x}{(\log 2)^2-6\log 2+13}$

(e) $\frac{1}{D+a}\phi(x)=e^{-ax}\int e^{ax}\phi(x)dx$

(f) $\frac{1}{f(D)}x^n\sin ax=\mathrm{Im}\,e^{ax}\frac{1}{f(D+a)}x^n$

Example: Solve $\left(D^2-5D+6\right)y=e^x\cos 2x$

Solution: $D^2-5D+6=0\Rightarrow (D-2)(D-3)=0\Rightarrow D=2,3\Rightarrow C.F.=C_1e^{2x}+C_2e^{3x}$

$$P.I.=\frac{1}{D^2-5D+6}e^x\cos 2x=e^x\frac{1}{(D+1)^2-5(D+1)+6}\cos 2x=e^x\frac{1}{D^2-3D+2}\cos 2x$$

$$\Rightarrow P.I.=e^x\frac{1}{-4-3D+2}\cos 2x=-e^x\frac{1}{3D+2}\cos 2x=-e^x\frac{3D-2}{9D^2-4}\cos 2x$$

$$\Rightarrow P.I.=-e^x\frac{3D-2}{9(-4)-4}\cos 2x=\frac{e^x}{40}(3D-2)\cos 2x$$

$$\Rightarrow P.I.=\frac{e^x}{40}(-6\sin 2x-2\cos 2x)=-\frac{e^x}{20}(3\sin 2x+\cos 2x)$$

Thus $y=C_1e^{2x}+C_2e^{3x}-\frac{e^x}{20}(3\sin 2x+\cos 2x)$

Example: Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\sin x$

Solution: $\left(D^2-2D+1\right)y=xe^x\sin x$

$\because \left(D^2-2D+1\right)=0\Rightarrow (D-1)^2=0\ D=1,1\ \Rightarrow C.F.=(c_1+c_2x)e^x$

$$P.I.=\frac{1}{(D-1)^2}e^x\sin x=e^x\frac{1}{(D+1-1)^2}.x\sin x=e^x\frac{1}{D^2}x\sin x=e^x\frac{1}{D}(-x\cos x+\sin x)$$

$$P.I.=e^x\left[-(x\sin x+\cos x)-\cos x\right]=-e^x\left[x\sin x+2\cos x\right]$$

$$\Rightarrow y=(c_1+c_2x)e^x-e^x\left[x\sin x+2\cos x\right]$$

MCQ (Multiple Choice Questions)

Q1. One of the possible solutions of the differential equation

$$y\sqrt{(1+x^2)}dy + x\sqrt{(1+y^2)}dx = 0$$ (where c is some constant) is

(a) $\left(\sqrt{1+y^2}\right)\left(\sqrt{1+x^2}\right) = c$

(b) $\dfrac{\sqrt{1+y^2}}{\sqrt{1+x^2}} = c$

(c) $\sqrt{1+y^2} + \sqrt{1+x^2} = c$

(d) $\sqrt{1+y^2} - \sqrt{1+x^2} = c$

Q2. The solution of the differential equation $x\dfrac{dy}{dx} + \cot y = 0$, subject to the initial condition $y = \dfrac{\pi}{4}$ at $x = \sqrt{2}$ is

(a) $x = 2\cos y$

(b) $x = 2\sec y$

(c) $x = 2\sin y$

(d) $x = 2\cos ecy$

Q3. The solutions to the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y+1}$ are a family of

(a) Circles with different radii

(b) Circles with different centres

(c) Straight lines with different slopes

(d) Straight lines with different intercepts on the y-axis

Q4. The solution of the differential equation $xy\dfrac{dy}{dx} = 3y^2 + x^2$ with the initial condition $y = 2$ when $x = 1$ is

(a) $2y^2 + x^2 = 9x^6$

(b) $y^2 + 2x^2 = 9x^6$

(c) $2y^2 + x^2 = 8x^6$

(d) $y^2 + 2x^2 = 8x^6$

Q5. The solution of the differential equation

$$(x+2y)(dx - dy) = dx + dy$$ (where a is some constant) is

(a) $3x + 3y + a = 2\log(3x + 6y - 1)$

(b) $3x - 3y + a = 2\log(3x + 6y - 1)$

(c) $3x - 3y + a = 2\log(3x - 6y - 1)$

(d) $3x + 3y + a = 2\log(3x - 6y - 1)$

Q6. For the differential equation $\frac{dy}{dx}+3y=e^{2x}$, the possible solution is:

(a) $y=c_1e^{2x}+c_2e^{3x}$ (b) $y=c_1e^{-2x}+c_2e^{3x}$

(c) $y=c_1e^{2x}+c_2e^{-3x}$ (d) $y=c_1e^{-2x}+c_2e^{-3x}$

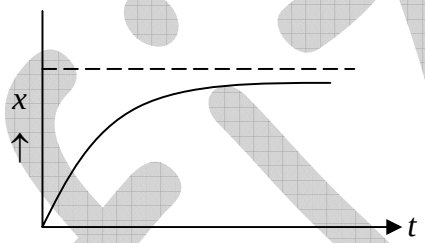
Q7. The solution of the differential equation for $y: x\frac{dy}{dx}+y=x^4$, subject to the initial condition $y=1$ at $x=1$ is

(a) $y=5x^4-4$ (b) $y=\frac{x^4}{5}+\frac{4x}{5}$

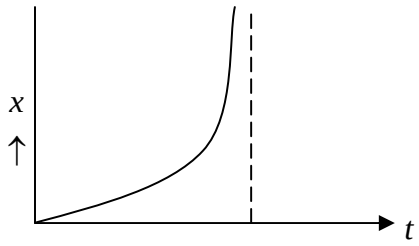
(c) $y=\frac{x^4}{5}+\frac{1}{5x}$ (d) $y=\frac{x^4}{5}+\frac{4}{5x}$

Q8. Which one of the following curves gives the solution of the differential equation $K_1\frac{dx}{dt}+K_2x=K_3$, where K_1, K_2 and K_3 are positive constants with initial conditions $x=0$ at $t=0$?

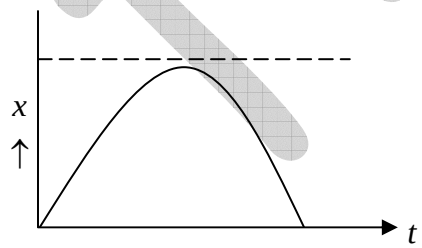
(a)



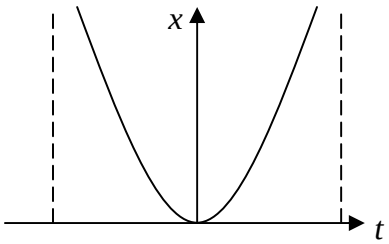
(b)



(c)



(d)



Q9. The solution of the differential equation $t\frac{dy}{dt}+y\log y=tye^t$

(a) $\log y=(t+1)e^t+c$ (b) $\log y=(t-1)e^t+c$

(c) $t\log y=(t+1)e^t+c$ (d) $t\log y=(t-1)e^t+c$

Q10. The solution of the differential equation

$$\left(1+y^2\right)dx=\left(\tan^{-1}y-x\right)dy$$ (where c is some constant) is

(a) $x=\left(\tan^{-1}y+2\right)+ce^{-\tan^{-1}y}$

(b) $x=\left(\tan^{-1}y-2\right)+ce^{-\tan^{-1}y}$

(c) $x=\left(\tan^{-1}y+1\right)+ce^{-\tan^{-1}y}$

(d) $x=\left(\tan^{-1}y-1\right)+ce^{-\tan^{-1}y}$

Q11. The solution of the differential equation

$$y\log y\frac{dx}{dy}+x-\log y=0$$ (where c is some constant) is

(a) $x\log y=\frac{1}{2}\left(\log y\right)^2+c$

(b) $x\log y=2\left(\log y\right)^2+c$

(c) $x\log y=\frac{1}{2}\left(\log y\right)^3+c$

(d) $y\log x=\frac{1}{2}\left(\log y\right)^2+c$

Q12. The solution of the differential equation

$$\left(e^y+2\right)\sin x dx-e^y\cos x dy=0$$ (where c is some constant) is

(a) $\left(e^y+2\right)\sin x=c$

(b) $\left(e^y+2\right)\cos x=c$

(c) $\left(e^y+2\right)\cos ecx=c$

(d) $\left(e^y+2\right)\sec x=c$

Q13. For the differential equation $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$ one of the possible solution (where c is some constant) is:

(a) $x+\frac{2}{y^2}+y^2=c$

(b) $x\left(y+\frac{2}{y^2}\right)=c$

(c) $x\left(y+\frac{2}{y^2}\right)+y^2=c$

(d) $x\left(1+\frac{2}{y^2}\right)+x^2=c$

Q14. The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2}-\frac{3dy}{dt}+2y=e^{3t}$ is

(a) $c_1e^t+c_2e^{2t}+\frac{1}{2}e^{3t}$

(b) $c_1e^{-t}+c_2e^{2t}+\frac{1}{2}e^{3t}$

(c) $c_1e^t+c_2e^{2t}+e^{3t}$

(d) $c_1e^{-t}+c_2e^{-2t}+e^{3t}$

Q15. The solution of the differential equation for $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+101y=10.4e^x$, subject to the initial conditions $y(0)=1.1$ and $\left.\frac{dy}{dx}\right|_{x=0}=-0.9$, is

(a) $e^{-x}\cos 10x+0.1e^x$ (b) $e^{-x}\sin 10x+0.1e^x$

(c) $e^{x}\cos 10x+0.1e^{-x}$ (d) $e^{x}\sin 10x+0.1e^{-x}$

Q16. The solution of the differential equation $\frac{d^2y}{dt^2}+4y=\cos 2t$ is:

(a) $A\cos 2t+B\sin 2t$ (b) $\frac{t\sin 2t}{4}$

(c) $A\cos 2t+B\sin 2t+\frac{t\sin 2t}{4}$ (d) $A\cos 2t+B\sin 2t+\frac{t^2\sin 2t}{4}$

Q17. The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2}-y=2\cosh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$, is

(a) $\frac{1}{2}\cos h(t)+t\sin h(t)$ (b) $-\sin h(t)+t\cos h(t)$

(c) $t\cos h(t)$ (d) $t\sin h(t)$

Q18. The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2}-y=2\sinh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$, is

(a) $\frac{1}{2}\cos h(t)+t\sin h(t)$ (b) $-\sin h(t)+t\cos h(t)$

(c) $t\cos h(t)$ (d) $t\sin h(t)$

NAT(Numerical Answer Type Questions)

Q19. The solution of the differential equation $\frac{dx}{dt} = x^2$ with the initial condition $x(0) = 1$ will blow up as t tends to…….

Q20. The value of λ ….., for which the differential equation $(xy^2 + \lambda x^2 y)dx + (x + y)x^2 dy = 0$ is exact

Q21. The particular integral of the differential equation $(D^2 + D + 1)y = \cos 2x$ is $\frac{1}{\alpha}(2\sin 2x - 3\cos 2x)$. Then the value of α is…..

Q22. The particular integral of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x$ is $\frac{1}{5}(\sin x - \alpha \cos x)$. Then the value of α is……..

Q23. The particular integral of the differential equation $(D^2 - 5D + 6)y = e^t \cos 2t$ is $-\frac{e^t}{20}(\alpha \sin 2t + \cos 2t)$. Then the value of α is……..

Q24. The particular integral of the differential equation $(D^2 - 4D + 4)y = x^3 e^{2x}$ is $e^{2x}\frac{x^\alpha}{20}$. Then the value of α is……..

Q25. The particular integral of the differential equation $(D^2 + 5D + 4)y = 3 - 2x$ is $\frac{1}{8}[\alpha - 4x]$. Then the value of α is……..

Q26. The maximum value of the solution $y(t)$…….of the differential equation for $y(t) + \ddot{y}(t) = 0$, subject to the initial conditions $\dot{y}(0) = 1$ and $y(0) = 1$ for $t \geq 0$, is

Q27. If the characteristic equation $\frac{d^2y}{dx^2} + 2\alpha\frac{dx}{dt} + y = 0$ has two equal roots, then the value of α ……. is

Q28. Consider the differential equation $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$. Given $x(0) = 20$ and $x(1) = \frac{10}{e}$, where $e = 2.718$, the value of $x(2)$ is ……

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: $\int \frac{y}{\sqrt{1+y^2}}dy + \int \frac{x}{\sqrt{1+x^2}}dx = c \Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} = c$

Ans. 2: (a)

Solution: $xdy = -\cot y dx \Rightarrow \tan y dy = -\frac{dx}{x}$

$\Rightarrow \int \tan y\, dy = -\int \frac{dx}{x} + \log A \Rightarrow \log \sec y = -\log x + \log A \Rightarrow \log \sec y = \log \frac{A}{x}$

$\Rightarrow x = A\cos y \Rightarrow \sqrt{2} = A\cos\frac{\pi}{4} \Rightarrow A = 2 \Rightarrow x = 2\cos y$

Ans. 3: (a)

Solution: $\frac{dy}{dx} = -\frac{x}{y+1} \Rightarrow xdx + ydy + dy = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + y = C_1 \Rightarrow x^2 + y^2 + 2y = 2C_1$

$\Rightarrow (x-0)^2 + (y+1)^2 = 2C_1 + 1 = C$

which is family of circles with different radii.

Ans. 4: (a)

Solution: $xy\frac{dy}{dx} = 3y^2 + x^2 \Rightarrow \frac{dy}{dx} = \frac{3y^2 + x^2}{xy}$ which is a homogeneous equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = \frac{3v^2x^2 + x^2}{vx^2} = \frac{3v^2+1}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{3v^2+1}{v} - v = \frac{2v^2+1}{v} \Rightarrow \frac{v}{2v^2+1}dv = \frac{dx}{x}$

$\Rightarrow \frac{1}{4}\ln(2v^2+1) = \ln x + \ln c' \Rightarrow \ln(2v^2+1) = \ln x^4 + \ln c \Rightarrow (2v^2+1) = cx^4$

$\Rightarrow \left(2\frac{y^2}{x^2} + 1\right) = cx^4$

$y=2$ when $x=1 \Rightarrow \left(2\frac{4}{1}+1\right)=c.1 \Rightarrow c=9 \Rightarrow 2y^2+x^2=9x^6$

Ans. 5: (b)

Solution: $\frac{dy}{dx}=\frac{x+2y-1}{x+2y+1}$. Let $x+2y=z \Rightarrow 1+2\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dz}{dx}=\frac{3z-1}{z+1}$

$\Rightarrow \int\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{3z-1}\right)dz=\int dx+c \Rightarrow \frac{z}{3}+\frac{4}{9}\log(3z-1)=x+c$

$\Rightarrow 3x-3y+a=2\log(3x+6y-1)$

Ans. 6: (c)

Solution: $\text{I.F.}=e^{\int 3dx}=e^{3x} \Rightarrow y\times e^{3x}=\int e^{2x}\times e^{3x}dx+c \Rightarrow y\times e^{3x}=\frac{e^{5x}}{5}+c \Rightarrow y=c_1e^{2x}+c_2e^{-3x}$

Ans. 7: (d)

Solution: $\frac{dy}{dx}+\frac{1}{x}y=x^3 \Rightarrow \text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$\Rightarrow y\times x=\int x^3\times xdx+c \Rightarrow y\times x=\frac{x^5}{5}+c \Rightarrow y=\frac{x^4}{5}+\frac{c}{x}$

Since $y=1$ at $x=1 \Rightarrow 1=\frac{1}{5}+\frac{c}{1} \Rightarrow c=\frac{4}{5} \Rightarrow y=\frac{x^4}{5}+\frac{4}{5x}$

Ans. 8: (a)

Solution: $\frac{dx}{dt}+\frac{K_2}{K_1}x=\frac{K_3}{K_1} \Rightarrow \frac{dx}{dt}+Ax=B \Rightarrow I.F.=e^{\int Adt}=e^{At} \Rightarrow x\times e^{At}=\int\left(B\times e^{At}\right)dt+c$

$\Rightarrow x\times e^{At}=B\times\frac{e^{At}}{A}+c \Rightarrow x=c_1+ce^{-At}$

Since $x=0$ at $t=0 \Rightarrow c_1+c=0 \Rightarrow c_1=-c \Rightarrow x=c\left(1-e^{-At}\right)$

Ans. 9: (d)

Solution: Divide by $yt \Rightarrow \frac{1}{y}\frac{dy}{dt}+\frac{1}{t}\log y=e^t$, put $\log y=z \Rightarrow \frac{1}{y}\frac{dy}{dt}=\frac{dz}{dt}$

$\Rightarrow \frac{dz}{dt}+\frac{1}{t}\cdot z=e^t \Rightarrow I\cdot F=e^{\int\frac{1}{t}dt}=e^{\log t}=t$

$\Rightarrow zt = \int te^t dt + c \Rightarrow t\log y = te^t - e^t + c$

Ans. 10: (d)

Solution: $\left(1+y^2\right)dx = \left(\tan^{-1} y - x\right)dy \Rightarrow \dfrac{dx}{dy} = \dfrac{\tan^{-1} y - x}{1+y^2}$

$\Rightarrow \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1} y}{1+y^2}$. This is a linear differential equation.

$I.F. = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Its solution is $x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \dfrac{\tan^{-1} y}{1+y^2} dy + c$

Put $\tan^{-1} y = t$ on R.H.S. so that $\dfrac{1}{1+y^2} dy = dt$

$x.e^{\tan^{-1} y} = \int e^t . t\, dt + C = t.e^t - e^t + c = e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) + c$

$\Rightarrow x = \left(\tan^{-1} y - 1\right) + ce^{-\tan^{-1} y}$

Ans. 11: (a)

Solution: $y\log y \dfrac{dx}{dy} + x - \log y = 0 \Rightarrow \dfrac{dx}{dy} + \dfrac{x}{y\log y} = \dfrac{1}{y}$

$I.F. = e^{\int \frac{1}{y\log y} dy} = e^{\log(\log y)} = \log y$

Its solution is $x.\log y = \int \dfrac{1}{y}(\log y) dy + c \Rightarrow x.\log y = \dfrac{1}{2}(\log y)^2 + c$.

Ans. 12: (b)

Solution: $M = \left(e^y + 2\right)\sin x, N = -e^y \cos x \Rightarrow \dfrac{\partial M}{\partial y} = e^y \sin x, \dfrac{\partial N}{\partial x} = e^y \sin x$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow \int \left(e^y + 2\right)\sin x dx + 0 = c' \Rightarrow \left(e^y + 2\right)\cos x = -c' = c$

Ans. 13: (c)

Solution: $M=\left(y^4+2y\right),\ N=\left(xy^3+2y^4-4x\right)$

$$\Rightarrow \frac{\partial M}{\partial y}=4y^3+2,\ \frac{\partial N}{\partial x}=y^3-4 \Rightarrow \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=-\frac{3}{y}=f(y)$$

then $\text{I.}F.=e^{\int -\frac{3}{y}dy}=e^{-3\log y}=\frac{1}{y^3}$

Multiplying by $\frac{1}{y^3}$ we get $\frac{1}{y^3}\left(y^4+2y\right)dx+\frac{1}{y^3}\left(xy^3+2y^4-4x\right)dy=0$ which is an exact equation. $\Rightarrow \int\left(y+\frac{2}{y^2}\right)dx+\int(2y)dy=C \Rightarrow x\left(y+\frac{2}{y^2}\right)+y^2=C$

Ans. 14: (a)

Solution: $\left(D^2-3D+2\right)=0 \Rightarrow D=1,\ 2 \Rightarrow C.F.=c_1e^t+c_2e^{2t}$

$$P.I.=\frac{1}{\left(D^2-3D+2\right)}e^{3t}=\frac{1}{\left(3^2-3\times 3+2\right)}e^{3t}=\frac{1}{2}e^{3t}$$

Ans. 15: (a)

Solution: $\left(D^2+2D+101\right)=0 \Rightarrow D=\frac{-2\pm\sqrt{4-4\times 101}}{2}=\frac{-2\pm 20i}{2}=-1\pm 10i$

$$C.F.=e^{-x}\left(A\cos 10x+B\sin 10x\right);$$

$$P.I.=\frac{1}{\left(D^2+2D+101\right)}10.4e^x=\frac{1}{\left(1+2\times 1+101\right)}10.4e^x=\frac{10.4}{104}e^x=0.1e^x$$

$$\Rightarrow y=e^{-x}\left(A\cos 10x+B\sin 10x\right)+0.1e^x$$

$$\because y(0)=1.1 \Rightarrow 0=1(A+0)+0.1=1.1 \Rightarrow A=1.$$

$$\Rightarrow y=e^{-x}\left(\cos 10x+B\sin 10x\right)+0.1e^x$$

$$\Rightarrow \frac{dy}{dx}=e^{-x}\left(-10\sin 10x+10B\cos 10x\right)-e^{-x}\left(\cos 10x+B\sin 10x\right)+0.1e^x$$

$$\because \left.\frac{dy}{dx}\right|_{x=0}=-0.9 \Rightarrow -0.9=1(-0+10B)-1(1+0)+0.1 \Rightarrow 10B=1-0.1-0.9 \Rightarrow B=0.$$

$\Rightarrow y = e^{-x}\cos 10x + 0.1e^{x}$

Ans. 16: (c)

Solution: $\left(D^2+4\right)=0 \Rightarrow D=+2i,-2i \Rightarrow C.F.=A\cos 2t + B\sin 2t$

Ans. 17: (d)

Solution: $\left(D^2-1\right)=0 \Rightarrow D=+1,-1 \Rightarrow C.F.=c_1e^{t}+c_2e^{-t}$

$$P.I.=\frac{1}{\left(D^2-1\right)}2\left(\frac{e^{t}+e^{-t}}{2}\right)=\frac{1}{\left(D^2-1\right)}\left(e^{t}+e^{-t}\right)=t\frac{1}{2D}\left(e^{t}+e^{-t}\right)=\frac{t}{2}\left(e^{t}-e^{-t}\right)$$

$$\Rightarrow y(t)=c_1e^{t}+c_2e^{-t}+\frac{t}{2}\left(e^{t}-e^{-t}\right)$$

$y(0)=0 \Rightarrow c_1+c_2=0$;

$$\frac{dy}{dt}=c_1e^{t}-c_2e^{-t}+\frac{1}{2}\left(e^{t}-e^{-t}\right)+\frac{t}{2}\left(e^{t}+e^{-t}\right),\left.\frac{dy}{dt}\right|_{t=0}=0 \Rightarrow c_1-c_2=0$$

$$\text{Thus } c_1=c_2=0 \Rightarrow y(t)=\frac{t}{2}\left(e^{t}-e^{-t}\right)=t\sinh(t)$$

Ans. 18: (c)

Solution: $\left(D^2-1\right)=0 \Rightarrow D=+1,-1 \Rightarrow C.F.=c_1e^{t}+c_2e^{-t}$

$$P.I.=\frac{1}{\left(D^2-1\right)}2\left(\frac{e^{t}-e^{-t}}{2}\right)=\frac{1}{\left(D^2-1\right)}\left(e^{t}-e^{-t}\right)=t\frac{1}{2D}\left(e^{t}-e^{-t}\right)=\frac{t}{2}\left(e^{t}+e^{-t}\right)$$

$$\Rightarrow y(t)=c_1e^{t}+c_2e^{-t}+\frac{t}{2}\left(e^{t}+e^{-t}\right)$$

$y(0)=0 \Rightarrow c_1+c_2=0$;

$$\frac{dy}{dt}=c_1e^{t}-c_2e^{-t}+\frac{1}{2}\left(e^{t}+e^{-t}\right)+\frac{t}{2}\left(e^{t}-e^{-t}\right) \Rightarrow \left.\frac{dy}{dt}\right|_{t=0}=0 \Rightarrow c_1-c_2=0$$

$$\text{Thus } c_1=c_2=0 \Rightarrow y(t)=\frac{t}{2}\left(e^{t}+e^{-t}\right)=t\cosh(t)$$

NAT (Numerical Answer Type Questions)

Ans. 19: 1

Solution: $\frac{dx}{dt} = x^2 \Rightarrow \int \frac{dx}{x^2} = \int dt \Rightarrow \frac{x^{-2+1}}{-2+1} = t + C \Rightarrow \frac{-1}{x} = t + C$

$\Rightarrow x(0) = 1 \Rightarrow \frac{-1}{1} = 0 + C \Rightarrow C = -1 \Rightarrow \frac{-1}{x} = t - 1 \Rightarrow x = \frac{1}{1-t}$ as $t \to 1$, x blows up.

Ans. 20: 3

Solution: $M = \left(xy^2 + \lambda x^2 y\right), N = (x+y)x^2 dy \Rightarrow \frac{\partial M}{\partial y} = 2xy + \lambda x^2, \frac{\partial N}{\partial x} = 3x^2 + 2xy$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \lambda = 3$

Ans. 21: 13

Solution: $P.I. = \frac{1}{D^2 + D + 1}.\cos 2x = \frac{1}{-2^2 + D + 1}.\cos 2x = \frac{1}{D-3}.\cos 2x.$

$\Rightarrow P.I. = \frac{D+3}{D^2 - 9}.\cos 2x = \frac{D+3}{-2^2 - 9}.\cos 2x = \frac{1}{13}(2\sin 2x - 3\cos 2x)$

Ans. 22: 2

Solution: $P \cdot I. = \frac{1}{D^2 + 2D + 2}\sin x = \frac{1}{-1 + 2D + 2}\sin x = \frac{2D-1}{4D^2 - 1}\sin x = -\frac{1}{5}(2D-1)\sin x$

$\Rightarrow P \cdot I. = -\frac{1}{5}(2\cos x - \sin x) = \frac{1}{5}(\sin x - 2\cos x)$

Ans. 23: 3

Solution: $P \cdot I \cdot = \frac{1}{D^2 - 5D + 6}e^t \cos 2t = e^t \frac{1}{(D+1)^2 - 5(D+1) + 6}\cos 2t$

$P \cdot I \cdot = e^t \frac{1}{D^2 - 3D + 2}\cos 2t = e^t \frac{1}{-4 - 3D + 2}\cos 2t$

$P \cdot I \cdot = -e^t \frac{1}{3D + 2}\cos 2t = -e^t \frac{3D-2}{9D^2 - 4}\cos 2t$

$P \cdot I \cdot = -e^t \frac{3D-2}{9 \times -4 - 4}\cos 2t = \frac{e^t}{40}(3D-2)\cos 2t = -\frac{e^t}{20}(3\sin 2t + \cos 2t)$

Ans. 24: 5

Solution: $P.I. = \frac{1}{D^2-4D+4}.x^3e^{2x} = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}.x^3.$

$\Rightarrow P.I. = e^{2x}\frac{1}{D^2}.x^3 = e^{2x}\frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x}\frac{x^5}{20}$

Ans25: 11

Solution: $P.I. = \left[D^2+5D+4\right]^{-1}(3-2x) = \frac{1}{4}\left[1+\frac{5}{4}D+\frac{5}{4}D^2\right]^{-1}(3-2x)$

$\Rightarrow P.I. = \frac{1}{4}\left[1-\frac{5}{4}D-\frac{5}{4}D^2\right](3-2x) = \frac{1}{4}\left[3-2x-\frac{5}{4}\times -2\right] = \frac{1}{8}[11-4x]$

Ans26: 1.41

Solution: $(D^2+1)=0 \Rightarrow D=\pm i \Rightarrow y = c_1e^{ix}+c_2e^{-ix} = A\cos x + B\sin x$

$\because y(0)=1 \Rightarrow 1 = A\times 1 + B\times 0 \Rightarrow A=1 \Rightarrow \dot{y} = -A\sin x + B\cos x$

$\Rightarrow \dot{y}(0)=1 \Rightarrow 1 = -A\times 0 + B\times 1 \Rightarrow B=1 \Rightarrow y = \cos x + \sin x$

For maxima, $y' = -\sin x + \cos x = 0 \Rightarrow \sin x = \cos x \Rightarrow x = 45^0$

$y'' = -\cos x - \sin x, \quad y'' < 0 \text{ for } x = 45^0$

$\Rightarrow y(\max) = \cos 45^0 + \sin 45^0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Ans27: 1

Solution: $(D^2+2\alpha D+1)=0 \Rightarrow m_1, m_1 = \frac{-2\alpha \pm \sqrt{4\alpha^2-4}}{2} \qquad \because m_1 = m_1 \Rightarrow \alpha = 1$

Ans. 28: 0.8566

Solution: $(D^2+3D+2)=0 \Rightarrow D=-1,-2 \Rightarrow x(t) = C_1e^{-t}+C_2e^{-2t}$

$\Rightarrow x(1) = \frac{10}{e} = C_1e^{-1}+C_2e^{-2} \Rightarrow C_1 + C_2e^{-1} = 10 \text{ and } C_1+C_2 = 20$

$\Rightarrow C_1 = \frac{10e-20}{e-1}; \; C_2 = \frac{10e}{e-1}$

$x(2) = \left(\frac{10e-20}{e-1}\right)e^{-2} + \left(\frac{10e}{e-1}\right)e^{-4} = 0.8566$